Časopis pro pěstování matematiky

Alois Švec

Cubic forms on Riemannian surfaces

Časopis pro pěstování matematiky, Vol. 103 (1978), No. 2, 181--185

Persistent URL: `http://dml.cz/dmlcz/108624`

Časopis pro pěstování matematiky, roč. 103 (1978), Praha

CUBIC FORMS ON RIEMANNIAN SURFACES

ALOIS ŠVEC, Olomouc

(Received August 31, 1976)

The purpose of this paper is to produce an invariant J associated with a cubic form Φ on a Riemannian positively curved 2-manifold M with the property that its positiveness on M closed ensures the vanishing of Φ.

Let M be a two-dimensional manifold endowed with a Riemannian metric ds^2. In a suitable domain $D \subset M$, we may choose a coframe of 1-forms (ω^1, ω^2) on D such that

(1) $$ds^2 = (\omega^1)^2 + (\omega^2)^2 .$$

Then there is exactly one 1-form ω_1^2 on D satisfying

(2) $$d\omega^1 = -\omega^2 \wedge \omega_1^2, \quad d\omega^2 = \omega^1 \wedge \omega_1^2 ;$$

the Gauss curvature K of ds^2 is then given by

(3) $$d\omega_1^2 = -K\omega^1 \wedge \omega^2 .$$

Choosing another coframe (τ^1, τ^2) on D with the property

(4) $$ds^2 = (\tau^1)^2 + (\tau^2)^2 ,$$

we have

(5) $$\omega^1 = \cos\alpha . \tau^1 - \sin\alpha . \tau^2 , \quad \omega^2 = \varepsilon \sin\alpha . \tau^1 + \varepsilon \cos\alpha . \tau^2 ; \quad \varepsilon = \pm 1 ;$$

(6) $$\omega_1^2 = \varepsilon(\tau_1^2 - d\alpha) .$$

Let Φ be a cubic differential form on M; in D, it may be written as

(7) $$\Phi = P(\omega^1)^3 + 3\,Q(\omega^1)^2\,\omega^2 + 3R\,\omega^1(\omega^2)^2 + S(\omega^2)^3$$

with respect to the coframe (ω^1, ω^2). Let us write, with respect to the coframe (τ^1, τ^2),

(8) $$\Phi = P^*(\tau^1)^3 + 3\,Q^*(\tau^1)^2\,\tau^2 + 3R^*\,\tau^1(\tau^2)^2 + S^*(\tau^2)^3 .$$

Then

$$(9)\qquad \begin{aligned} P^* &= \cos^3\alpha\,.\,P + 3\varepsilon\sin\alpha\cos^2\alpha\,.\,Q + 3\sin^2\alpha\cos\alpha\,.\,R + \varepsilon\sin^3\alpha\,.\,S\,,\\ Q^* &= -\sin\alpha\cos^2\alpha\,.\,P + \varepsilon\cos\alpha(\cos^2\alpha - 2\sin^2\alpha)\,.\,Q +\\ &\quad + \sin\alpha(2\cos^2\alpha - \sin^2\alpha)\,.\,R + \varepsilon\sin^2\alpha\cos\alpha\,.\,S\,,\\ R^* &= \sin^2\alpha\cos\alpha\,.\,P + \varepsilon\sin\alpha(\sin^2\alpha - 2\cos^2\alpha)\,.\,Q +\\ &\quad + \cos\alpha(\cos^2\alpha - 2\sin^2\alpha)\,.\,R + \varepsilon\sin\alpha\cos^2\alpha\,.\,S\,,\\ S^* &= -\sin^3\alpha\,.\,P + 3\varepsilon\sin^2\alpha\cos\alpha\,.\,Q - 3\sin\alpha\cos^2\alpha\,.\,R + \varepsilon\cos^3\alpha\,.\,S\,. \end{aligned}$$

Let us introduce the covariant derivatives $P_1, \ldots, S_2$ of $P, \ldots, S$ with respect to the coframe (ω^1, ω^2) by means of

$$(10)\qquad \begin{aligned} dP - 3Q\omega_1^2 &= P_1\omega^1 + P_2\omega^2\,,\\ dQ + (P - 2R)\,\omega_1^2 &= Q_1\omega^1 + Q_2\omega^2\,,\\ dR + (2Q - S)\,\omega_1^2 &= R_1\omega^1 + R_2\omega^2\,,\\ dS + 3R\omega_1^2 &= S_1\omega^1 + S_2\omega^2\,. \end{aligned}$$

Then

$$(11)\qquad \begin{aligned} &\{dP_1 - (P_2 + 3Q_1)\,\omega_1^2\} \wedge \omega^1 + \{dP_2 + (P_1 - 3Q_2)\,\omega_1^2\} \wedge \omega^2 =\\ &\quad = 3QK\omega^1 \wedge \omega^2\,,\\ &\{dQ_1 + (P_1 - Q_2 - 2R_1)\,\omega_1^2\} \wedge \omega^1 +\\ &\quad + \{dQ_2 + (P_2 + Q_1 - 2R_2)\,\omega_1^2\} \wedge \omega^2 = (2R - P)\,K\omega^1 \wedge \omega^2\,,\\ &\{dR_1 + (2Q_1 - R_2 - S_1)\,\omega_1^2\} \wedge \omega^1 +\\ &\quad + \{dR_2 + (2Q_2 + R_1 - S_2)\,\omega_1^2\} \wedge \omega^2 = (S - 2Q)\,K\omega^1 \wedge \omega^2\,,\\ &\{dS_1 + (3R_1 - S_2)\,\omega_1^2\} \wedge \omega^1 +\\ &\quad + \{dS_2 + (3R_2 + S_1)\,\omega_1^2\} \wedge \omega^2 = -3RK\omega^1 \wedge \omega^2\,, \end{aligned}$$

and we get the existence of the second order covariant derivatives $P_{11}, \ldots, S_{22}$ such that

$$(12)\qquad \begin{aligned} dP_1 - (P_2 + 3Q_1)\,\omega_1^2 &= P_{11}\omega^1 + (P_{12} - \tfrac{3}{2}QK)\,\omega^2\,,\\ dP_2 + (P_1 - 3Q_2)\,\omega_1^2 &= (P_{12} + \tfrac{3}{2}QK)\,\omega^1 + P_{22}\omega^2\,,\\ dQ_1 + (P_1 - Q_2 - 2R_1)\,\omega_1^2 &= Q_{11}\omega^1 + (Q_{22} + PK)\,\omega^2\,,\\ dQ_2 + (P_2 + Q_1 - 2R_2)\,\omega_1^2 &= (Q_{12} + 2RK)\,\omega^1 + Q_{22}\omega^2\,,\\ dR_1 + (2Q_1 - R_2 - S_1)\,\omega_1^2 &= R_{11}\omega^1 + (R_{12} + 2QK)\,\omega^2\,, \end{aligned}$$

$$dR_2 + (2Q_2 + R_1 - S_2)\,\omega_1^2 = (R_{12} + SK)\,\omega^1 + R_{22}\omega^2\,,$$

$$dS_1 + (3R_1 - S_2)\,\omega_1^2 = S_{11}\omega^1 + (S_{12} + \tfrac{3}{2}RK)\,\omega^2\,,$$

$$dS_2 + (3R_2 + S_1)\,\omega_1^2 = (S_{12} - \tfrac{3}{2}RK)\,\omega^1 + S_{22}\omega^2\,.$$

Let $P_1^*, \ldots, S_2^*$ be the covariant derivatives of $P^*, \ldots, S^*$ with respect to the coframe (τ^1, τ^2). Then

(13)
$$\begin{aligned}
P_1^* = {} & \cos^4\alpha \,.\, P_1 + \varepsilon \sin\alpha\cos^3\alpha \,.\, P_2 + 3\varepsilon\sin\alpha\cos^3\alpha \,.\, Q_1 + \\
& + 3\sin^2\alpha\cos^2\alpha \,.\, Q_2 + 3\sin^2\alpha\cos^2\alpha \,.\, R_1 + \\
& + 3\varepsilon\sin^3\alpha\cos\alpha \,.\, R_2 + \varepsilon\sin^3\alpha\cos\alpha \,.\, S_1 + \sin^4\alpha \,.\, S_2\,, \\
P_2^* = {} & - \sin\alpha\cos^3\alpha \,.\, P_1 + \varepsilon\cos^4\alpha \,.\, P_2 - 3\varepsilon\sin^2\alpha\cos^2\alpha\, Q_1 + \\
& + 3\sin\alpha\cos^3\alpha \,.\, Q_2 - 3\sin^3\alpha\cos\alpha \,.\, R_1 + \\
& + 3\varepsilon\sin^2\alpha\cos^2\alpha \,.\, R_2 - \varepsilon\sin^4\alpha \,.\, S_1 + \sin^3\alpha\cos\alpha \,.\, S_2\,, \\
Q_1^* = {} & - \sin\alpha\cos^3\alpha \,.\, P_1 - \varepsilon\sin^2\alpha\cos^2\alpha \,.\, P_2 + \\
& + \varepsilon\cos^2\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, Q_1 + \\
& + \sin\alpha\cos\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, Q_2 + \\
& + \sin\alpha\cos\alpha(2\cos^2\alpha - \sin^2\alpha) \,.\, R_1 + \\
& + \varepsilon\sin^2\alpha(2\cos^2\alpha - \sin^2\alpha) \,.\, R_2 + \varepsilon\sin^2\alpha\cos^2\alpha \,.\, S_1 + \\
& + \sin^3\alpha\cos\alpha \,.\, S_2\,, \\
Q_2^* = {} & \sin^2\alpha\cos^2\alpha \,.\, P_1 - \varepsilon\sin\alpha\cos^3\alpha \,.\, P_2 - \\
& - \varepsilon\sin\alpha\cos\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, Q_1 + \\
& + \cos^2\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, Q_2 - \sin^2\alpha(2\cos^2\alpha - \sin^2\alpha) \,.\, R_1 + \\
& + \varepsilon\sin\alpha\cos\alpha(2\cos^2\alpha - \sin^2\alpha) \,.\, R_2 - \\
& - \varepsilon\sin^3\alpha\cos\alpha \,.\, S_1 + \sin^2\alpha\cos^2\alpha \,.\, S_2\,, \\
R_1^* = {} & \sin^2\alpha\cos^2\alpha \,.\, P_1 + \varepsilon\sin^3\alpha\cos\alpha \,.\, P_2 + \\
& + \varepsilon\sin\alpha\cos\alpha(\sin^2\alpha - 2\cos^2\alpha) \,.\, Q_1 + \\
& + \sin^2\alpha(\sin^2\alpha - 2\cos^2\alpha) \,.\, Q_2 + \cos^2\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, R_1 + \\
& + \varepsilon\sin\alpha\cos\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, R_2 + \\
& + \varepsilon\sin\alpha\cos^3\alpha \,.\, S_1 + \sin^2\alpha\cos^2\alpha \,.\, S_2\,, \\
R_2^* = {} & - \sin^3\alpha\cos\alpha \,.\, P_1 + \varepsilon\sin^2\alpha\cos^2\alpha \,.\, P_2 - \\
& - \varepsilon\sin^2\alpha(\sin^2\alpha - 2\cos^2\alpha) \,.\, Q_1 +
\end{aligned}$$

$$+ \sin\alpha\cos\alpha(\sin^2\alpha - 2\cos^2\alpha) \,.\, Q_2 -$$
$$- \sin\alpha\cos\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, R_1 +$$
$$+ \varepsilon\cos^2\alpha(\cos^2\alpha - 2\sin^2\alpha) \,.\, R_2 -$$
$$- \varepsilon\sin^2\alpha\cos^2\alpha \,.\, S_1 + \sin\alpha\cos^3\alpha \,.\, S_2 \,,$$

$$S_1^* = - \sin^3\alpha\cos\alpha \,.\, P_1 - \varepsilon\sin^4\alpha \,.\, P_2 + 3\varepsilon\sin^2\alpha\cos^2\alpha \,.\, Q_1 +$$
$$+ 3\sin^3\alpha\cos\alpha \,.\, Q_2 - 3\sin\alpha\cos^3\alpha \,.\, R_1 -$$
$$- 3\varepsilon\sin^2\alpha\cos^2\alpha \,.\, R_2 + \varepsilon\cos^4\alpha \,.\, S_1 + \sin\alpha\cos^3\alpha \,.\, S_2 \,,$$

$$S_2^* = \sin^4\alpha \,.\, P_1 - \varepsilon\sin^3\alpha\cos\alpha \,.\, P_2 - 3\varepsilon\sin^3\alpha\cos\alpha \,.\, Q_1 +$$
$$+ 3\sin^2\alpha\cos^2\alpha \,.\, Q_2 + 3\sin^2\alpha\cos^2\alpha \,.\, R_1 -$$
$$- 3\varepsilon\sin\alpha\cos^3\alpha \,.\, R_2 - \varepsilon\sin\alpha\cos^3\alpha \,.\, S_1 + \cos^4\alpha \,.\, S_2 \,.$$

By means of (9) + (13) and (10) + (12), it is easy to prove

Lemma. *In D, consider the 1-form*

$$\tau = \{(Q + \tfrac{2}{3}S)\,P_1 - PQ_1 + SR_1 - (R + \tfrac{2}{3}P)\,S_1\}\,\omega^1 + \tag{14}$$
$$+ \{(Q + \tfrac{2}{3}S)\,P_2 - PQ_2 + SR_2 - (R + \tfrac{2}{3}P)\,S_2\}\,\omega^2$$

and the function

$$J = P_2Q_1 - P_1Q_2 + R_2S_1 - R_1S_2 + \tfrac{4}{3}(P_2S_1 - P_1S_2)\,. \tag{15}$$

Then, in the obvious notation,

$$\tau^* = \varepsilon\tau\,, \quad J^* = J\,. \tag{16}$$

Further,

$$\mathrm{d}\tau = \{2J + (P^2 + 3Q^2 + 3R^2 + S^2)\,K\}\,\omega^1 \wedge \omega^2\,. \tag{17}$$

Thus J is an invariant of Φ on M, and τ is globally defined on the orientable M. From the Stokes theorem, we get the following

Theorem. *Let M be an orientable two-dimensional Riemannian manifold with a positive Gauss curvature endowed with a cubic differential form Φ. Let J be the above introduced invariant associated with Φ. If $\Phi \equiv 0$ on the boundary ∂M of M and $J \geqq 0$ on M, then $\Phi \equiv 0$ on M.*

It remains to clarify the geometric interpretation of the invariant J.

The associated Euclidean connection on M is determined by means of the formulas

$$\nabla m = \omega^1 v_1 + \omega^2 v_2\,, \quad \nabla v_1 = -\omega_1^2 v_2\,, \quad \nabla v_2 = \omega_1^2 v_1\,. \tag{18}$$

Let γ be a curve on M, and let

(19) $$v = xv_1 + yv_2$$

be a tangent vector field along γ. Because of

(20) $$\nabla v = (\mathrm{d}x - y\omega_1^2)\, v_1 + (\mathrm{d}y + x\omega_1^2)\, v_2\,,$$

v is parallel along γ if and only if

(21) $$\mathrm{d}x = y\omega_1^2\,, \quad \mathrm{d}y = -x_1^2\omega \text{ along } \gamma\,.$$

Now, let us choose $m_0 \in M$ and $v_0, w_0 \in T_{m_0}(M)$; let

(22) $$w_0 = \xi v_1 + \eta v_2\,.$$

Further, choose a curve $\gamma \subset M$ going through m_0 and having w_0 for its tangent vector at m_0 and a parallel vector field v (19) along γ such that $v(m_0) = v_0$. Then

(23) $$w_0\, \Phi(v) = (P_1x^3 + 3Q_1x^2y + 3R_1xy^2 + S_1y^3)\, \xi + + (P_2x^3 + 3Q_2x^2y + 3R_2xy^2 + S_2y^3)\, \eta$$

does not depend on γ. Thus we are in the position to define, for each vector $v_0 \in \in T_{m_0}(M)$, the 1-form φ_{v_0} by means of

(24) $$\varphi_{v_0}(w_0) = w_0\, \Phi(v)\,; \quad w_0 \in T_{m_0}(M)\,.$$

At m_0, choose an orthonormal frame; let the coframe (ω^1, ω^2) be chosen in such a way that the dual frame (v_1, v_2) at m_0 is exactly our frame. Then

(25) $$\varphi_{v_1} = P_1\omega^1 + P_2\omega^2\,, \quad \varphi_{v_2} = S_1\omega^1 + S_2\omega^2\,,$$

$$\varphi_{v_1+v_2} = (P_1 + 3Q_1 + 3R_1 + S_1)\, \omega^1 + (P_2 + 3Q_2 + 3R_2 + S_2)\, \omega^2\,,$$

$$\varphi_{v_1-v_2} = (P_1 - 3Q_1 + 3R_1 - S_1)\, \omega^1 + (P_2 - 3Q_2 + 3R_2 - S_2)\, \omega^2$$

and

(26) $$(\varphi_{v_1+v_2} + \varphi_{v_1-v_2}) \wedge \varphi_{v_1} - (\varphi_{v_1+v_2} + \varphi_{v_1-v_2}) \wedge \varphi_{v_2} = 6J\, \mathrm{d}o\,,$$

$\mathrm{d}o = \omega^1 \wedge \omega^2$ being the area element of M.

Author's address: **771 46 Olomouc, Leninova 26, (Přírodovědecká fakulta UP).**