Ján Jakubík
Lattice ordered groups with cyclic linearly ordered subgroups

Časopis pro pěstování matematiky, Vol. 101 (1976), No. 1, 88--90

Persistent URL: http://dml.cz/dmlcz/108691

Terms of use:

© Institute of Mathematics AS CR, 1976

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
LATTICE ORDERED GROUPS
WITH CYCLIC LINEARLY ORDERED SUBGROUPS

Ján Jakubík, Košice

(Received October 18, 1974)

In this note a solution is given to a problem proposed by Conrad and Montgomery [3] on lattice ordered groups $G$ with the property that each linearly ordered subgroup of $G$ is cyclic.

Let $G$ be an archimedean lattice ordered group. Consider the following conditions for $G$:

(a) $G$ is singular;
(b) each linearly ordered subgroup of $G$ is cyclic.

In [3] it was proved that (a) implies (b) while the problem whether (a) is implied by (b) remained open. We shall show that the answer is negative in general; nonetheless, (b) \implies (a) is valid if $G$ is complete.

For the basic notions and notations cf. Birkhoff [1] and Fuchs [4]. Let $G$ be a lattice ordered group. An element $0 \leq g \in G$ is called singular, if $x \wedge (g - x) = 0$ for each $x \in G$ with $0 \leq x \leq g$. It is easy to verify that a strictly positive element $g \in G$ is singular if and only if the interval $[0, g]$ is a Boolean algebra. The $l$-group $G$ is singular, if for each $0 < g \in G$ there is a singular element $h \in G$ such that $0 < h \leq g$.

Singular lattice ordered groups were investigated in the papers [2], [5], [6], [7], [8].

The following theorem is known (cf. [2]):

(A) Let $G$ be a complete $l$-group. Then there are $l$-subgroups $A$, $B$ of $C$ such that $A$ is singular, $B$ is a vector lattice and $G = A \times B$.

(The symbol $A \times B$ denotes the direct sum of $l$-groups $A$ and $B$.)

Now let $G$ be a complete $l$-group that is not singular. According to (A) we have $B \neq \{0\}$ and hence there is $b, 0 < b \in B$. Let $R$ be the set of all reals; since $B$ is a vector lattice, for each $r \in R$ there exists $rb \in B$. Denote $B_1 = \{rb : r \in R\}$. Then $B_1$ is a linearly ordered subgroup of $G$ that fails to be cyclic. Therefore (a) is implied by (b) whenever $G$ is a complete lattice ordered group.

The following example shows that an archimedean lattice ordered group fulfilling (b) need not be singular.

88
Let \( Q \) be the set of all rational numbers and let \( G_0 \) be the set of all real functions defined on \( Q \). For \( f, g \in G_0 \) we put \( f \leq g \) if \( f(x) \leq g(x) \) for all \( x \in Q \). Then \((G_0; +, \leq)\) is an archimedean lattice ordered group. Let \( \phi \) be a one-to-one mapping of the set \( N \) of all positive integers onto the set \( Q \). Further, let \( G \) be the set of all \( f \in G_0 \) with the following properties:

(i) \( 2^{n-1} f(\varphi(n)) \) is an integer for all \( n \in N \);
(ii) there are irrational numbers \( \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_m < \beta_m \) such that \( f \) is a constant on each set \( Q \cap [\alpha_i, \beta_i] \) (\( i = 1, \ldots, m \)) and \( f(x) = 0 \) for each \( x \in Q \setminus \bigcup [\alpha_i, \beta_i] \) (\( i = 1, \ldots, m \)).

Then \( (G_0; -F, g) \) is an archimedean lattice ordered group.

Let \( \varphi \) be a one-to-one mapping of the set \( N \) of all positive integers onto the set \( Q \). Further, let \( G \) be the set of all \( f \in G_0 \) with the following properties:

(i) \( 2^{n-1} f(\varphi(n)) \) is an integer for all \( n \in N \);
(ii) there are irrational numbers \( \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \ldots \leq \alpha_m < \beta_m \) such that \( f \) is a constant on each set \( Q \cap [\alpha_i, \beta_i] \) (\( i = 1, \ldots, m \)) and \( f(x) = 0 \) for each \( x \in Q \setminus \bigcup [\alpha_i, \beta_i] \) (\( i = 1, \ldots, m \)).

Then \( G \) is an \( l \)-subgroup of \( G_0 \).

Let \( H = \{0\} \) be a linearly ordered subgroup of \( G \). For each \( h \in H \) put

\[ s(h) = \{ x \in Q : h(x) \neq 0 \}. \]

**Lemma 1.** Let \( 0 \neq h_i \in H \) (\( i = 1, 2 \)). Then \( s(h_1) = s(h_2) \).

**Proof.** Suppose that \( s(h_1) \neq s(h_2) \). Then we can assume that there is \( x \in s(h_1) \setminus s(h_2) \). We have \( |h_i| \in H, s(|h_i|) = s(h_i) \) (\( i = 1, 2 \)). The elements \( |h_1|, |h_2| \) are comparable and \( |h_1|(x) > 0 = |h_2|(x) \). Since \( h_2 \neq 0 \), there is \( y \in s(h_2) \) and hence \( |h_2|(y) > 0 \). There is a positive integer \( n \) with \( n|h_2|(y) > |h_1|(y) \). Since \( n|h_2| \in H \), the elements \( n|h_2| \) and \( |h_1| \) are comparable, thus \( n|h_2| > |h_1| \). But

\[ 0 = n|h_2|(x) < |h_1|(x) \]

and this is a contradiction.

For \( x \in Q \) let

\[ F_x = \{ h(x) : h \in G \}. \]

Obviously \( F_x \) is an additive group.

**Lemma 2.** Let \( 0 \neq h_0 \in H, x \in s(h_0) \). The mapping

\[ \varphi_1 : h \rightarrow h(x) \]

is an isomorphism of \( H \) into \( F_x \).

**Proof.** If \( h_1, h_2 \in H \) and \( \circ \in \{ +, \wedge, \vee \} \), then

\[ \varphi_1(h_1 \circ h_2) = h_1(x) \circ h_2(x), \]

thus \( \varphi_1 \) is a homomorphism of \( H \) into \( F_x \). Let \( \varphi_1(h_1) = \varphi_1(h_2) \) and suppose that \( h_1 \neq h_2 \). Then \( h = h_1 - h_2 \in H, h \neq 0 \) and \( h(x) = 0 \neq h_0(x) \). Thus \( s(h) \neq s(h_0) \), which contradicts Lemma 1. Therefore \( h_1 = h_2 \) and hence \( \varphi_1 \) is an isomorphism.

**Lemma 3.** The \( l \)-group \( H \) is cyclic.

**Proof.** Let \( x \in Q, \varphi^{-1}(x) = n \). There exist irrational numbers \( \alpha, \beta \) such that \( x \in [\alpha, \beta] \) and \( \varphi^{-1}(y) \geq n \) for each \( y \in [\alpha, \beta] \cap Q \). Let \( f \in G_0 \) such that \( f(z) = 2^{1-n} \ldots \)
for each \( z \in [\alpha, \beta] \cap Q \) and \( f(z) = 0 \) otherwise. Then \( f \in G_0 \) and hence \( 2^{1-n} \in F_x \).

Thus by (i), \( 2^{1-n} \) is a generator of the group \( F_x \) and therefore \( F_x \) is cyclic. Hence each subgroup of \( F_x \) is cyclic; by Lemma 2, \( H \) is cyclic.

**Lemma 4.** Let \( 0 < f \in G_0 \). Then \( f \) is not singular.

**Proof.** Suppose that \( f \) is singular. Then each \( f_1 \in G_0, 0 < f_1 < f \) is singular. There exist irrational numbers \( \alpha, \beta \) and a real \( c \neq 0 \) such that \( f(x) = c \) for each \( x \in [\alpha, \beta] \cap Q \). Let \( f_1 \in G_0 \) such that \( f_1(x) = f(x) = c \) for each \( x \in Q \cap [\alpha, \beta] \) and \( f_1(x) = 0 \) otherwise. Clearly \( f_1 \in G \) and \( 0 < f_1 \leq f \). Let

\[
N_1 = \{ \varphi^{-1}(x) : x \in Q \cap [\alpha, \beta] \}.
\]

Let \( k \) be the least element of \( N_1 \). According to (i) and (ii), \( 2^{k-1}c \) is an integer. We can choose irrational numbers \( \alpha < \beta \) such that \( [\alpha, \beta] \subset [\alpha_1, \beta_1] \) and \( \varphi(k) \notin [\alpha, \beta] \).

Let \( y \in [\alpha, \beta] \cap Q \). Put \( \varphi^{-1}(y) = t \). Since \( t > k \), we infer that \( 2^{k-1}(\frac{1}{c}) \) is an integer. Thus the function \( g \in G_0 \) defined by

\[
g(x) = \frac{1}{c} \quad \text{if} \quad x \in [\alpha, \beta] \cap Q \quad \text{and} \quad g(x) = 0 \quad \text{otherwise}
\]

belongs to \( G_0 \). We have \( 0 < 2g < f_1 \), hence \( g < f_1 - g \) and therefore

\[
g \wedge (f_1 - g) = g > 0;
\]

thus \( f_1 \) cannot be singular. This shows that \( f \) is not singular.

From Lemma 2 and Lemma 4 it follows that there exists an archimedean lattice ordered group fulfilling (b) with no singular elements.

**References**


**Author’s address:** 040 01 Košice, Švermova 5 (Vysoké učení technické).