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A THEOREM ON 2-CONNECTED GRAPHS

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Let G be a graph. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of G, respectively. The number of elements of $V(G)$ is referred to as the order of G. We say that vertices r and s of G are independent if they are distinct and non-adjacent. If $u \in V(G)$, then by $\deg_G u$ we denote the degree of the vertex u in G.

We say that a connected graph G of order $p \geqq 3$ is 2-connected if for every $v \in V(G)$, the graph $G - v$ is connected. The terms not defined here can be found in BEHZAD and CHARTRAND [1].

We shall say that a vertex w of a 2-connected graph G is *weak* if $G - w$ is 2-connected. Theorem 2.10 in [1], due to A. KAUGARS, can be reformulated as follows: Every 2-connected graph contains either a weak vertex or a vertex of degree 2. We shall prove the following stronger result:

Theorem. *Every 2-connected graph of order $p \geqq 4$ contains either a pair of adjacent weak vertices or a pair of independent vertices of degree* 2.

Proof. The case $p = 4$ is obvious. Assume that $p = n \geqq 5$ and that the statement is proved for $4 \leqq p \leqq n - 1$. Let G be a 2-connected graph of order p. Assume that G contains no pair of adjacent weak vertices. We shall consider the following two possibilities:

(1) For every edge $x = u_0u_1$ of G, at least one of the vertices u_0 and u_1 is adjacent to a vertex of degree 2. Then G contains a vertex r_1 of degree 2 which is adjacent to distinct vertices r_0 and r_2. Assume that G contains no pair of independent vertices of degree 2. Then without loss of generality we can assume that $\deg_G r_0 \geqq 3$. If $\deg_G r_2 \geqq 3$, then by r we denote the vertex r_2; if $\deg_G r_2 = 2$, then by r we denote the vertex adjacent to r_2 and different from r_1. Obviously, $\deg_G r \geqq 3$. It is easily seen that if $s_1, s_2 \in V(G) - \{r_0, r_1, r_2, r\}$, then the vertices s_1 and s_2 are non-adjacent. As no vertex in the set $V(G) - \{r_0, r_1, r_2, r\}$ has degree 2, G contains a vertex of degree 1, which is a contradiction. This means that G contains a vertex s_1 of degree 2 such that the vertices r_1 and s_1 are independent.

(2) There is an edge $y = uu'$ of G such that neither u nor u' is adjacent to a vertex of degree 2. Without loss of generality we can assume that the vertex u is not a weak one. Thus $G - u$ is not 2-connected and there is a vertex v such that the graph $G - u - v$ is disconnected. It is easily seen that there exist subgraphs F_1 and F_2 of G such that $V(F_1) \cup V(F_2) = V(G)$, $V(F_1) \cap V(F_2) = \{u, v\}$, $3 \leqq |V(F_1)| \leqq |V(F_2)|$, $E(F_1) \cup E(F_2) = E(G)$, and $E(F_1) \cap E(F_2) = \emptyset$. As u is adjacent to no vertex of degree 2, $|V(F_1)| \geqq 4$. Hence $|V(G)| \geqq 6$.

Let $i \in \{1, 2\}$. We construct a graph G_i as follows: (a) if $\deg_{F_i} u = 1 = \deg_{F_i} v$, then $V(G_i) = V(F_i)$ and $E(G_i) = E(F_i) \cup \{uv\}$; (b) if either $\deg_{F_i} u > 1$ or $\deg_{F_i} v > 1$, then $V(G_i) = V(F_i) \cup \{w_i\}$ and $E(G_i) = E(F_i) \cup \{uw_i, vw_i\}$, where w_i is a vertex different from the vertices of F_i. Clearly, G_i is 2-connected. It is easily seen that for every vertex $t \in V(F_i)$, t is a weak vertex of G_i if and only if it is a weak vertex of G. This means that G_i contains no pair of adjacent weak vertices. As $5 \leqq |V(G_i| \leqq p - 1$, G_i contains a pair of independent vertices of degree 2. There is a vertex $t_i \in V(F_i) - \{u, v\}$ such that $\deg_{G_i} t_i = 2$. Obviously, $\deg_G t_i = 2$. As t_1 and t_2 are independent vertices of G, the proof is complete.

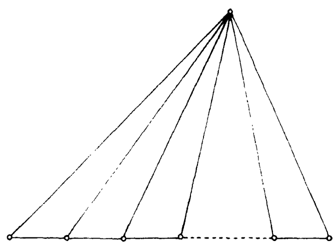

Fig. 1.

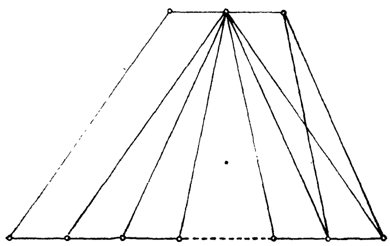

Fig. 2.

Remark. As follows from Fig. 1, for every integer $p \geqq 4$, there is a 2-connected graph of order p such that (i) it contains a pair of independent vertices of degree 2, (ii) it contains precisely two weak vertices, and (iii) the weak vertices are independent. As follows from Fig. 2, for every integer $p \geqq 6$, there is a 2-connected graph of order p such that (i) it contains a pair of adjacent weak vertices, (ii) it contains precisely two vertices of degree 2, and (iii) the vertices of degree 2 are adjacent.

Reference

[1] *M. Behzad* and *G. Chartrand:* Introduction to the Theory of Graphs. Allyn and Bacon, Inc., Boston 1971.

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