Norman Levinson

Criteria for the limit-point case for second order linear differential operators

Časopis pro pěstování matematiky a fysiky, Vol. 74 (1949), No. 1, 17--20

Persistent URL: http://dml.cz/dmlcz/109150

CRITERIA FOR THE LIMIT-POINT CASE FOR SECOND ORDER LINEAR DIFFERENTIAL OPERATORS.

By NORMAN LEVINSON.*)

(Received November 9, 1948.)

It is an important result of WEYL that a differential operator $\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$, where $p(x)$ and $q(x)$ are real and continuous and $p(x) > 0$ for large x, falls into one of two cases, the limit-point case or the limit-circle case. In the limit-circle case all solutions of

$$(pu')' + (q + \lambda) u = 0, \tag{1}$$

where λ is any complex constant, satisfy

$$\int^{\infty} |u(x)|^2 \, dx < \infty. \tag{2}$$

In the limit point case at most one independent solution of (1) satisfies (2). (When $Im\lambda \neq 0$ then exactly one such solution exists satisfying (2).) If it is shown for any particular value of λ, in particular $\lambda = 0$, that two independent solutions cannot satisfy (2) then it follows that the operator is in the limit-point case.

WINTNER and HARTMAN**) have recently given certain sufficient criteria for the limit-point case when $p(x) \equiv 1$. They concern themselves with

$$u'' + f(x) u = 0, \tag{3}$$

where $f(x)$ is continuous for large x and have proved that if $f(x)$ is bounded from above, that is if there exists some positive constant K such that

$$f(x) < K, \tag{4}$$

then at most one solution of (3) satisfies (2), i. e. the limit-point case.

*) JOHN SIMON GUGGENHEIM Memorial Fellow on leave from Massachusetts Institute of Technology.

) PHILLIP HARTMAN and AUREL WINTNER, Criteria of Non-Degeneracy for the Wave Equation, American Journal of Mathematics, vol **70 (1948), pp. 295—308, where other references are given.

They also show that a sufficient criterion for the limit-point case is

$$f(x_2) - f(x_1) < K(x_2 - x_1), \; x_2 > x_1. \tag{5}$$

If $f(x)$ is monotone increasing and satisfies

$$\int^{\infty} \frac{\mathrm{d}x}{\sqrt{f(x)}} = \infty \tag{6}$$

then again the limit-point case prevails. There is a very considerable gap between criteria (4) and (6) which we shall show can be narrowed very considerably. In fact we shall show

Theorem I. *If for large* x

$$f(x) < Kx^2 \tag{7}$$

then (3) *cannot have two independent solutions satisfying* (2), i. e. (3) *is in the limit-point case.*

Since (5) implies (but is not implied by) the weaker condition $f(x) < Kx$ we see that (7) includes (5) as a special case and of course (7) includes (4). Note that (7) is a one-sided condition and of course requires no monotonicity for $f(x)$.

The condition (7) is again slightly weaker than the condition

Theorem II. *If* $m(x)$ *is a positive monotone non-decreasing function of* x *such that*

$$\int^{\infty} \frac{\mathrm{d}x}{(m(x))^{\frac{1}{2}}} = \infty, \tag{8}$$

$$\overline{\lim_{x\to\infty}} \frac{m'(x)}{(m(x))^{\frac{3}{2}}} < \infty, \tag{9}$$

and if for large x

$$f(x) < K\, m(x) \tag{10}$$

then (3) *is in the limit-point case.*

Since we can take $m(x) = x^2$ in Theorem II we see that Theorem I is a consequence of Theorem II. Note that (10) is again a one-sided condition and monotonicity for $f(x)$ is not at all required.

We turn now to (1) where we do not require $p(x) \equiv 1$. Here we have

Theorem III. *The equation*

$$(pu')' + qu = 0 \tag{11}$$

cannot have two independent solutions satisfying (2) *if for large* x

$$q(x) < K \tag{12}$$

and

$$\int^{\infty} \frac{\mathrm{d}x}{(p(x))^{\frac{1}{2}}} = \infty. \tag{13}$$

Under rather wide conditions it follows from applying standard transformations (due to Liouville) to (11) that (13) is a best possible condition. (Briefly if (13) does not hold (11) can be transformed to a regular second order differential equation on a finite interval where of course all solutions are integrable squared.)

Theorem III is a special case of

Theorem IV. *The equation* (11) *is in the limit-point case if there exists a positive monotone non-decreasing function $M(x)$ such that for large x*

$$q(x) < K\, M(x), \tag{14}$$

$$\int^{\infty} \frac{\mathrm{d}x}{(p(x)\, M(x))^{\frac{1}{2}}} = \infty, \tag{15}$$

and

$$\overline{\lim_{x\to\infty}}\, \frac{(p(x))^{\frac{1}{2}}\, M'(x)}{(M(x))^{\frac{3}{2}}} < \infty. \tag{16}$$

In the special case $M(x) \equiv 1$, Theorem IV yields Theorem III. In the special case $p(x) \equiv 1$, Theorem IV yields Theorem II. Thus we see that we have only to prove Theorem IV which we now do. Since p and q are real we can restrict our considerations to real solutions of (11).

From (11) we have

$$\frac{qu^2}{M} = \frac{-(pu')'\,u}{M}.$$

Integrating from some convenient point $x = a$ we have for $x > a$

$$\int_a^x \frac{qu^2}{M}\,\mathrm{d}x = -\frac{puu'}{M}\Big]_a^x + \int_a^x \frac{p(u')^2}{M}\,\mathrm{d}x - \int_a^x \frac{puu'M'}{M^2}\,\mathrm{d}x.$$

Let us assume (2) holds. Then by (14) we have that there exists a K_1, such that

$$K_1 > -\frac{p(x)\,u(x)\,u'(x)}{M(x)} + \int_a^x \frac{p(u')^2}{M}\,\mathrm{d}x - \int_a^x \frac{puu'M'}{M^2}\,\mathrm{d}x. \tag{17}$$

Now let us assume that the first integral on the right in (17) diverges. Then

$$H(x) = \int_a^x \frac{p(u')^2}{M}\,\mathrm{d}x$$

is a positive monotone increasing function tending to infinity. Using (16) and the Schwartz inequality we see that there exist constants K_2 and K_3 such that

$$\left|\int_a^x \frac{puu'M'}{M^2}\,dx\right| < K_2 \int_a^x \left|\left(\frac{p(x)}{M(x)}\right)^{\frac{1}{2}} uu'\right| dx$$

$$< K_3 \left(\int_a^x \frac{p(x)\,(u')^2}{M(x)}\,dx\right)^{\frac{1}{2}} = K_3\, H^{\frac{1}{2}}(x).$$

In (17) this yields

$$K_1 > H(x) - \frac{p(x)\,u(x)\,u'(x)}{M(x)} - K_3\, H^{\frac{1}{2}}(x).$$

Since $H(x) \to \infty$ we see that the above inequality implies that for large x

$$\frac{p(x)\,u(x)\,u'(x)}{M(x)} > \tfrac{1}{2}H(x).$$

Thus for large x, $u(x)$ and $u'(x)$ have the same sign. Thus $|u(x)|$ is monotone increasing and (2) cannot hold. We see then that if $u(x)$ satisfies (2) then $H(x)$ remains finite, that is

$$\int_a^\infty \frac{p(x)\,(u')^2}{M(x)}\,dx < \infty. \tag{18}$$

We now use a device of WINTNER. Two independent solutions of (11), $u_1(x)$ and $u_2(x)$ satisfy $p(x)(u_1u'_2 - u_2u'_1) = c$, where c is a constant and is not zero. Or

$$\left(\frac{p(x)}{M(x)}\right)^{\frac{1}{2}} u_2'(x)\,u_1(x) - \left(\frac{p(x)}{M(x)}\right)^{\frac{1}{2}} u_1'(x)\,u_2(x) = \frac{c}{(p(x)\,M(x))^{\frac{1}{2}}}. \tag{19}$$

Suppose u_1 and u_2 satisfy (2). Then they also satisfy (18). Thus the left side of (19) is integrable over (a, ∞). By (15) the right side of (19) is not integrable over (a, ∞). Thus we arrive at a contradiction and establish Theorem IV.

Mathematics Institute, Copenhagen.

Kriteria pro případ „limitního bodu“ u lineárních diferenciálních operátorů 2. řádu.

(Obsah předešlého článku.)

Jde o podmínky postačující k tomu, aby rovnice (1) nemohla míti dva lineárně nezávislé integrály u, vyhovující podmínce (2). Nejobecnější podmínka je dána větou IV. Stačí, když existuje kladná neklesající funkce $M(x)$ taková, že platí (14) (pro velká x), (15) a (16). Toto kriterium je zostřením kriteria, jež nedávno podali HARTMAN a WINTNER.