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DIRECT PRODUCT DECOMPOSITIONS
OF PSEUDO MV–ALGEBRAS

JÁN JAKUBÍK

Dedicated to Professor František Šik
on the occasion of his 80. anniversary

Abstract. In this paper we deal with the relations between the direct product decompositions of a pseudo MV-algebra and the direct product decompositions of its underlying lattice.

I. Introduction

Direct product decompositions of MV-algebras have been investigated in [8]. It is well-known that for each MV-algebra \( A \) there exists an abelian lattice ordered group \( G \) with a strong unit \( u \) such that \( A \) can be obtained by a well-defined construction from \( G \); in accordance with the notation from the monograph by Cignoli, D’Ottaviano and Mundici [2] we write \( A = \Gamma(G, u) \). One of the items dealt with in [8] was the relation between direct product decompositions of \( A \) and direct product decompositions of \( G \).

The MV-algebra is an algebraic structure of type \((2, 1, 0, 0)\) (cf. [2]); the binary operation is denoted by the symbol \( \oplus \) and it is assumed to be commutative.

The notion of MV-algebra can be generalized in such a way that the assumption of the commutativity of the operation \( \oplus \) is omitted (cf. Georgescu and Iorgulescu [5], [6], and Rachůnek [11]).

The results of [6] were used by Dvurečenskij and Pulmannová [3]; further, the results of [11] were applied by Chajda, Halaš and Rachůnek [1].

For further results on pseudo MV-algebras cf. Dvurečenskij [4], Leustean [10], Rachůnek [12] and the author [9].

In the present paper we apply the terminology and the notation from [6]. Thus we deal with an algebra of type \((2, 1, 1, 0, 0)\) which is called a pseudo MV-algebra; for the definition, cf. Section 2 below. (I remark that the substantial part of this paper has been finished before I was acquainted with [11] and [1].)

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If $G$ is a lattice ordered group (which need not be abelian) and if $0 \leq u \in G$, then by similar construction as in the abelian case we can construct the algebraic structure $\mathcal{A} = \Gamma(G, u)$. It turns out that $\mathcal{A}$ is a pseudo $MV$-algebra.

In [3] it was proposed the problem whether for each pseudo $MV$-algebra $\mathcal{A}_1$ there exists a lattice ordered group $G_1$ with a strong unit $u_1$ such that $\mathcal{A}_1 = \Gamma(G_1, u_1)$. Dvurečenskij [4] proved that the answer is positive.

Let $\mathcal{A}$ be the underlying set of the pseudo $MV$-algebra $\mathcal{A}$. By applying the basic pseudo $MV$-operations we can define a partial order $\leq$ on the set $\mathcal{A}$ such that $L(\mathcal{A}) = (\mathcal{A}; \leq)$ is a bounded distributive lattice. (Cf. [6].)

2. Preliminaries

We start by recalling the definition of pseudo $MV$-algebra (cf. [6], [3]).

Let $\mathcal{A} = (A; \oplus, -, \sim, 0, 1)$ be an algebraic structure, where $A$ is a nonempty set, $\oplus$ is a binary operation, $-$ and $\sim$ are unary operations, 0 and 1 are nulary operations on $A$. For each $x, y \in A$ we put

$$y \odot x = (x^- \oplus y^-)^{\sim}.$$  

The algebraic structure $\mathcal{A}$ is called a pseudo $MV$-algebra if the following conditions are satisfied for each $x, y, z \in A$:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1 \sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^- \oplus y^-)^\sim$;
(A6) $x \oplus (x^- \odot y) = y \oplus (y^- \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$;
(A7) $x \odot (x^- \odot y) = (x \odot y^-) \odot y$;
(A8) $(x^-)^\sim = x$.

We define $x \leq y$ iff $x^- \oplus y = 1$. Then $\leq$ is a partial order on $A$. Put $L(\mathcal{A}) = (A; \leq)$. 


Proposition 2.1. (Cf. [6].) \( L(A) \) is a lattice with the least element 0 and with the greatest element 1. Moreover, for each \( x, y \in A \) we have
\[
x \lor y = x \oplus (x \sim \bowtie y), \quad x \land y = x \bowtie (x \sim \oplus y).
\]

Lemma 2.2 (Cf. [6]). If \( \{y_i\}_{i \in I} \subseteq A \) and if \( \bigvee_{i \in I} y_i \) exists in \( L(A) \), then the lattice \( L(A) \) satisfies the condition
\[
x \land \bigvee_{i \in I} y_i = \bigvee_{i \in I} (x \land y_i).
\]
Hence, in particular, \( L(A) \) is a distributive lattice.

Let \( G \) be a lattice ordered group and let \( 0 \leq u \in G \). The group operation in \( G \) is denoted by +, though we do not assume that this operation is commutative. Further, let \( A \) be the interval \([0, u]\) of \( G \). For \( x, y \in A \) we put
\[
x \oplus y = (x + y) \wedge u, \quad x \sim = u - x, \quad x \sim = -x + u.
\]

Proposition 2.3 (Cf. [6]). The algebraic structure \( \mathcal{A} = (A; \oplus, -, \sim, 0, u) \) is a pseudo MV-algebra.

If \( A \) is as in 2.3, then we denote \( A = \Gamma(G, u) \).

3. Auxiliary results

Again, let \( A \) be a pseudo MV-algebra and \( L = L(A) \). Assume that \( e \) is an element of \( A \) which has a complement \( e' \) in the lattice \( L(A) \). In view of 2.2, \( e' \) is uniquely determined.

Consider the intervals
\[
X_1 = [0, e], \quad X_2 = [0, e']
\]
of the lattice \( L \). For \( a \in A \) we put
\[
a_1 = e \land a, \quad a_2 = e' \land a.
\]

Lemma 3.1 (Cf. [6]). Let \( p, q \in A \), \( p \land q = 0 \). Then
\[
p \oplus q = p \lor q = q \oplus p.
\]

Corollary 3.2. Let \( x^1 \in X_1, x^2 \in X_2 \). Then \( x^1 \oplus x^2 = x^2 \oplus x^1 \).

Lemma 3.3. \( a = a_1 \oplus a_2 = a_1 \lor a_2 \) for each \( a \in A \).
Proof. The distributivity of $L$ yields
$$a = a \land 1 = a \land (e \lor e') = (a \land e) \lor (a \land e') = a_1 \lor a_2.$$ Hence in view of 3.1, $a = a_1 \oplus a_2$. □

Lemma 3.4. Let $a \in A$, $x^1 \in X_1$, $x^2 \in X_2$, $a = x^1 \oplus x^2$. Then $x^1 = a_1$ and $x^2 = a_2$.

Proof. By applying 3.1, 3.2 and the distributivity of $L$ we obtain
$$x^1 = x^1 \land (x^1 \lor x^2) = x^1 \land (x^1 \oplus x^2) = x^1 \land a = x^1 \land (a_1 \lor a_2) = (x^1 \land a_1) \lor (x^1 \land a_2).$$
We have $x^1 \land a_2 = 0$, whence $x^1 = x^1 \land a_1$ and thus $x^1 \leq a_1$. Similarly we verify that $a_1 \leq x^1$, thus $a_1 = x^1$. Analogously, $a_2 = x^2$. □

Lemma 3.5 (Cf. [6], Propos. 1.20). If $x \land y = 0$, then $x \land (y \oplus z) = x \land z$.

Lemma 3.6. The set $X_1$ is closed with respect to the operation $\oplus$.

Proof. Let $a, b \in X_1$. Then we have $a \land e' = 0$. Hence according to 3.5 we obtain
$$e' \land (a \oplus b) = e' \land b = 0,$$
whence $(a \oplus b)_2 = 0$. Thus 3.3 yields $(a \oplus b)_1 = a \oplus b$. Therefore $a \oplus b \in X_1$. □

Lemma 3.7. Let $a, b \in A$. Then
$$(a \oplus b)_1 = a_1 \oplus b_1, \quad (a \oplus b)_2 = a_2 \oplus b_2.$$ Proof. In view of 3.2 and 3.3 we have
$$a \oplus b = (a_1 \oplus a_2) \oplus (b_1 \oplus b_2) = a_1 \oplus (a_2 \oplus b_1) \oplus b_2 =$$
$$= a_1 \oplus (b_1 \oplus a_2) \oplus b_2 = (a_1 \oplus b_1) \oplus (a_2 \oplus b_2).$$
According to 3.6, $a_1 \oplus b_1 \in X_1$. Similarly, $a_2 \oplus b_2 \in X_2$. Thus 3.4 yields
$$(a \oplus b)_1 = a_1 \oplus b_1, \quad (a \oplus b)_2 = a_2 \oplus b_2.$$ □

Proposition 3.8 (Cf. [3], 4.4.3, Exercise 7.6.4.5). Let $A$ be a pseudo MV-algebra, $A \neq \{0\}$. Then there exists a lattice ordered group $G$ with an element $0 < u \in G$ such that $A$ can be embedded (as a pseudo MV-algebra) into the pseudo MV-algebra $\Gamma(G, u)$.

Let $x \in A$. Denote
$$P_x = \{p \in A : p \oplus x = 1\},$$
$$Q_x = \{q \in A : x \oplus q = 1\}.$$
Lemma 3.9. Let \( x \in A \). Then

\[
x^− = \min P_x, \quad x^\sim = \min Q_x.
\]

**Proof.** In view of 1.5 in [6] we have \( x^− \oplus x = 1 \), whence \( x^− \in P_x \). Let \( G \) and \( u \) be as in 3.8. Thus \( u = 1 \) and \( x^− + x = u \). Let \( p \in P_x \). Hence

\[
u = 1 = p \oplus x = (p + x) \land u,
\]

which yields

\[
p + x \geq u = x^− + x.
\]

Therefore \( p \geq x^− \). Hence \( x^− = \min P_x \). Analogously we verify that \( x^\sim = \min Q_x \). \(\square\)

**Remark 3.9.1.** From 3.9 we conclude that unary operations \(-\) and \(\sim\) are uniquely determined by the operation \(\oplus\) and by the partial order \(\leq\) on \(A\).

For \( x \in X_1 \) we denote

\[
P^1_x = \{p \in X_1 : p \oplus x = e\},
\]

\[
Q^1_x = \{q \in X_1 : x \oplus q = e\}.
\]

Lemma 3.10. Let \( x \in X_1 \). Then

\[
(x^−)_2 = (x^\sim)_2 = e',
\]

\[
(x^−)_1 = \min P^1_x, \quad (x^\sim)_1 = \min Q^1_x.
\]

**Proof.** Clearly \( (x^−)_2 \leq e' \). By way of contradiction, suppose that \( (x^−)_2 < e' \). Then

\[
1 = x^− \oplus x = (x^−)_1 \oplus (x^−)_2 \oplus x.
\]

Since \((x^−)_2 \in X_2\) and \(x \in X_1\), in view of 3.2 we have

\[
(x^−)_2 \oplus x = x \oplus (x^−)_2.
\]

Therefore

\[
1 = (x^−)_1 \oplus x \oplus (x^−)_2.
\]

In view of 3.6, \((x^−)_1 \oplus x \in X_1\). Thus \((x^−)_1 \oplus x \leq e\), whence

\[
1 \leq e \oplus (x^−)_2 = e \lor (x^−)_2.
\]

In view of the distributivity and according to the relation \((x^−)_2 < e'\) we get \(e \lor (x^−)_2 < 1\), which is a contradiction. Thus \((x^−)_2 = e'\).
Further, by way of contradiction, assume that the relation
\[(x^-)_1 = \min P^1_x\]
fails to hold.

According to 3.7 we have
\[(x^-)_1 \oplus x_1 = (x^- \oplus x)_1 = 1_1 = 1 \land e = e,\]
whence \((x^-)_1 \in P^1_x\). Thus in view of the assumption there is \(z \in P^1_x\) such that
\[(x^-)_1 \not\in z.\]
Denote
\[t = (x^-)_1 \land z.\]
Then \(t < (x^-)_1 \land z\). Then \(t < (x^-)_1\) and hence \(t \in X_1\), yielding \(t_1 = t\). We have
\[t \oplus x = ((x^-)_1 \land z) \oplus x = ((x^-)_1 \oplus x) \land (z \oplus x)\]
(in view of 1.16, [5]). Since \(z \oplus x = e\), we get \(t \oplus x = e\), whence \(t \in P^1_x\).
In view of the distributivity of \(L\) we obtain
\[t \oplus e' = t \lor e' < (x^-)_1 \lor e' = (x^-)_1 \lor (x^-)_2 = x^-.\]
Further,
\[(t \oplus e') \oplus x = t \oplus (e' \oplus x) = t \lor (x \lor e') = (t \lor x) \lor e' = e \lor e' = e \lor e' = 1.\]
Since \(t \oplus e' < x^-\), in view of 3.9 we arrived at a contradiction. Therefore \((x^-)_1 = \min P^1_x\).

The remaining relations concerning \(x^-\) can be proved analogously. \(\square\)

**Lemma 3.11.** Let \(x \in X_1\) and let \(G\) be as in 3.8. Let \(b^1 \in G\), \(b^1 + x = e\). Then \(b^1 = \min P^1_x\).

**Proof.** It suffices to apply analogous steps as in the proof of 3.9. \(\square\)

Analogously we have

**Lemma 3.11.1.** Let \(y \in X_2\) and let \(b^2 \in G\), \(y + b^2 = e'\). Then \(b^2 = \min Q^1_y\).

It is obvious that if \(b^1\) and \(b^2\) are as in 3.11 and 3.11.1, then \(b^1 \in X_1\) and \(b^2 \in X_2\).
Put \(b = b^1 \oplus b^2\). In view of 3.4 we have
\[b_1 = b^1, \quad b_2 = b^2.\]

**Lemma 3.12.** Let \(a \in A\). Denote \(a_1 = x\), \(a_2 = y\) and let \(b^1, b^2\) be as above. Then
\[(a^-)_1 = b^1.\]

**Proof.** We have \(b^1 \land b^2 = 0\). Hence \(b = b_1 + b_2\); similarly, \(a = a_1 + a_2\). Thus
\[b + a = (b_1 + b_2) + (a_1 + a_2) = b_1 + (b_2 + a_1) + a_2 = b_1 + (a_1 + b_2) + a_2 =\]
\[= (b_1 + a_1) + (b_2 + a_2) = e + e' = e \lor e' = 1.\]
Therefore \(b = a^-\). Hence
\[(a^-)_1 = b^1 = b^1.\] \(\square\)
4. The pseudo MV-algebra $X_1$

Assume that $\mathcal{A}$ is an MV-algebra and let $e, e', X_1$ and $X_2$ be as in the previous section.

We have already observed (cf. 3.6) that $X_1$ is closed with respect to the operation $\oplus$. Further, it is obvious that $X_1$ is also closed with respect to the operations $\land$ and $\lor$.

In view of 3.9 and 3.10 we define the unary operations $-^{(e)}$ and $\sim^{(e)}$ on $X_1$ by putting

$$x^{-(e)} = \min P^1_x, \quad x^\sim^{(e)} = \min Q^1_x$$

for each $x \in X_1$.

Now, we define a binary operation $\odot_e$ on $X_1$ by

$$y \odot_e x = (x^{-(e)} \oplus y^{-(e)})^{\sim(e)}.$$

Let us consider the algebraic structure

$$X_1 = (X_1; \oplus, -^{(e)}, \sim^{(e)}, 0, e).$$

For $a \in A$ let $a_1$ be as in Section 3; let us now apply the notation

$$a_1 = \phi_1(a).$$

Then the mapping $\phi_1 : A \to X_1$ is surjective. We have clearly $\phi_1(0) = 0, \phi_1(1) = e$. Moreover, in view of 3.7, $\phi_1$ is a homomorphism with respect to the operation $\oplus$.

For proving that $\phi_1$ is a homomorphism with respect to the operation $-^{(e)}$ we have to verify that the relation

$$\phi_1(a^{-}) = \phi_1(a)^{-(e)}$$

is valid for each $a \in A$.

Let $a \in A$. Denote $x = a_1$. Under the notation as in 3.12 we have

$$\phi_1(a^{-}) = (a^{-})_1 = b^1.$$

In view of 3.11, $b^1 = \min P^1_x$. Thus

$$b_1 = x^{-(e)} = a_1^{-^{(e)}} = \phi_1(a)^{-^{(e)}}.$$

Analogously we verify that $\phi_1$ is a homomorphism with respect to the operation $\sim$.

Summarizing, we have

**Lemma 4.1.** $\phi_1$ is a homomorphism of the pseudo MV-algebra $\mathcal{A}$ onto the algebraic structure $X_1$.

In view of 4.1 we conclude that $X_1$ satisfies all the identities (A1) - (A8) (under the notation modified in the obvious way). Hence we obtain
Corollary 4.2. \( X_1 \) is a pseudo MV-algebra.

Analogous consideration can be performed for \( X_2 \); we apply the symbols \( \varphi_2 \) and \( X_2 \).

For each \( a \in A \) we put

\[
\varphi(a) = (\varphi_1(a), \varphi_2(a)).
\]

The direct product of pseudo MV-algebras is defined in the usual way; cf. e.g., [5].

Proposition 4.3. \( \varphi \) is an isomorphism of the pseudo MV-algebra \( A \) onto the direct product \( A_1 \times A_2 \).

Proof. Under the notation as in Section 3 we have

\[
\varphi(a) = (a_1, a_2) = (a \land e, a \land e').
\]

Since \( L \) is a distributive lattice \( \varphi \) is a bijection. Then 4.1 yields that \( \varphi \) is an isomorphism. \( \square \)

Let \( B(A) \) be the set of all elements of \( L \) which have a complement. Then \( B(A) \) is a Boolean algebra; it was dealt with in [5], Section 4.

By a direct product decomposition of \( A \) we understand an isomorphism of \( A \) onto a direct product of pseudo MV-algebras.

In view of 4.3, to each element \( e \) of \( B(A) \) there corresponds a uniquely determined two-factor direct product decomposition of \( A \).

5. TWO-FACTOR DIRECT PRODUCT DECOMPOSITIONS OF \( A \)

In this section we show that each two-factor direct product decomposition of the pseudo MV-algebra \( A \) is constructed, up to isomorphism, by the method described in Section 4.

Assume that we have a two-factor direct product decomposition of \( A \), i.e., an isomorphism

\[
\psi : A \to A_1 \times A_2,
\]

such that \( \psi \) is a bijection.

Let \( L_1 \) and \( L_2 \) be the lattices corresponding to \( A_1 \) or to \( A_2 \), respectively. Since the lattice operations in \( L = L(A) \) are defined by means of the operations \( \oplus, \neg \) and \( \sim \), we conclude that the mapping

\[
\psi : L \to L_1 \times L_2
\]

determines a direct product decomposition of the lattice \( L \).

The lattices \( L_1 \) and \( L_2 \) must be bounded; let \( 0^i \) and \( 1^i \) be the least or the greatest element of \( L_i \), respectively \((i = 1, 2)\).
Put $A_1 \times A_2 = A_0$ and let $A_0$ be the underlying set of $A_0$. Further, let $A_i$ be the underlying set of $A_i (i = 1, 2)$. Denote

$$A_i^* = \{(a^1, 0^2) : a^1 \in A_1\}, \quad A_2^* = \{(0^1, a^2) : a^2 \in A_2\},$$

$$X_1 = \psi^{-1}(A_1^*), \quad X_2 = \psi^{-1}(A_2^*),$$

$$e = \psi^{-1}((1^1, 0^2)), \quad e' = \psi^{-1}((0^1, 1^2)).$$

Then we clearly have

**Lemma 5.1.** (i) $e \lor e' = 1$, $e \land e' = 0$; (ii) $X_1 = [0, e]$, $X_2 = [0, e']$.

Thus in view of 4.3 we have a direct product decomposition

$$\varphi : A \rightarrow X_1 \times X_2,$$

where

$$\varphi(a) = (e \land a, e' \land a)$$

for each $a \in A$.

Further, for each $a^1 \in A_1$ and each $a^2 \in A_2$ we put

$$\varphi_1(a^1) = (a^1, 0^2), \quad \varphi_2(a^2) = (0^1, a^2).$$

Both the mappings $\varphi_1 : A_1 \rightarrow A_1^*$ and $\varphi_2 : A_2 \rightarrow A_2^*$ are bijections. Therefore there exist pseudo $MV$-algebras $A_1^*$ and $A_2^*$ such that for $i \in \{1, 2\}$ we have

(i) $A_i^*$ is the underlying set of $A_i^*$;

(ii) $\varphi_i$ is an isomorphism of $A_i$ onto $A_i^*$.

For $i \in \{1, 2\}$ we denote by $\psi_i$ the mapping $\psi$ reduced to the set $X_i$.

In view of 3.6, the set $X_1$ is closed with respect to the operation $\oplus$; the same is valid for the set $X_2$. Further, according to 5.1, both $X_1$ and $X_2$ are closed with respect to the lattice operations $\lor$ and $\land$. From this we obtain

**Lemma 5.2.** Let $i \in \{1, 2\}$. Then $\psi_i$ is an isomorphism with respect to the operation $\oplus$ and with respect to the lattice operations $\lor, \land$.

Let $X_1$ and $X_2$ be as in (3). From 5.2 and 3.9.1 we conclude

**Lemma 5.3.** Let $i \in \{1, 2\}$. Then $\psi_i$ is an isomorphism of the pseudo $MV$-algebra $X_i$ onto the pseudo $MV$-algebra $A_i^*$.

For $i \in \{1, 2\}$ and $x^i \in X_i$ put

$$\psi_i^0(x^i) = \varphi_i^{-1}(\psi_i(x^i)) .$$

From the properties of $\varphi_i$ and from 5.3 we get

**Proposition 5.4.** Let $i \in \{1, 2\}$. Then $\psi_i^0$ is an isomorphism of the pseudo $MV$-algebra $X_i$ onto the pseudo $MV$-algebra $A_i$.

In other words, the direct factors standing in (1) are, up to isomorphism, the same as the direct factors standing in (3), and the direct product decomposition (3) is constructed by the procedure from Section 4.
Moreover, we show that the mapping $\psi$ is uniquely determined by the mappings $\varphi$, $\psi_1$ and $\psi_2$. In fact, let $a \in A$ and

$$\varphi(a) = (a_1, a_2), \quad \psi(a) = (a^1, a^2).$$

Since $a_1 \in X_1$, there is $p \in A_1$ with

$$\psi_1(a_1) = \psi(a_1) = (p, 0^2).$$

Similarly, there is $q \in A_2$ such that

$$\psi_2(a_2) = \psi(a_2) = (0^1, q).$$

Then we have

**Proposition 5.5.** Under the notation as above, $a^1 = p$ and $a^2 = q$.

**Proof.** In view of (4) we have $a_1 = a \wedge e$. Thus according to (2) we get

$$\psi(a_1) = \psi(a) \wedge \psi(e),$$

$$(p, 0^2) = (a_1, a_2) \wedge (1^1, 0^2) = (a^1, 0^2)$$

whence $p = a^1$. Similarly we obtain $q = a^2$. $\square$

6. **DIRECT PRODUCT DECOMPOSITIONS OF $L(A)$**

In the present section we apply the previous results for dealing with the direct product decompositions having an arbitrary number of direct factors.

We investigate the relations between the direct product decompositions of $A$ and those of $L(A)$.

**Theorem 6.1.** Suppose that

$$\varphi : A \to \prod_{i \in I} A_i$$

is a direct product decomposition of a pseudo MV-algebra $A$. Then, at the same time, we have a direct product decomposition

$$\varphi : L(A) \to \prod_{i \in I} L(A_i).$$

**Proof.** The underlying sets of $A$ and of $L(A)$ coincide; a similar situation occurs for $A_i$ and $L(A_i)$. Now it suffices to apply the fact that the operations $\lor$ and $\land$ are defined by means of the basic operations of MV-algebra $A$ (cf. 2.1). $\square$

Now let us assume that we are given a direct product decomposition of the lattice $L(A) = L$ of the form

(1) $$\psi_1 : L \to \prod_{i \in I} L_i.$$ If there exists a direct product decomposition

(2) $$\psi_2 : A \to \prod_{i \in I} A_i$$
such that $\psi_2 = \psi_1$ and $L(A_i) = L_i$ for each $i \in I$, then we say the direct product decomposition (1) induces the direct product decomposition (2).

The following assertion is obvious.
Lemma 6.2. Let $A$ be a pseudo MV-algebra, $L = L(A)$. Further, let $L'$ be a lattice and let $\varphi$ be an isomorphism of $L$ onto $L'$. Then there exists a pseudo MV-algebra $A'$ with $L(A') = L'$ such that $\varphi$ is an isomorphism of $A$ onto $A'$.

Let $A$ and $L(A)$ be as above. Assume that we have a direct product decomposition

$$\psi : L \to \prod_{i \in I} L_i.$$  

For $i \in I$ we denote by $0^i$ and $1^i$ the least and the greatest element of $L_i$, respectively.

There exist elements $e_i$ and $e'_i$ in $L$ such that

$$\begin{align*}
\psi(e_i)_i &= 1^i, \quad \psi(e_i)_j = 0^j \quad \text{for } j \in I \setminus \{i\}, \\
\psi(e'_i)_i &= 0^i, \quad \psi(e'_i)_j = 1^j \quad \text{for } j \in I \setminus \{i\}.
\end{align*}$$

Then we have

$$e_i \land e'_i = 0, \quad e_i \lor e'_i = 1.$$  

Hence in view of 4.3 there exists a direct product decomposition

$$\varphi_i : A \to X_{i1} \times X_{i2},$$

where (under the usual notation) we have

$$X_{i1} = [0, e_i], \quad X_{i2} = [0, e'_i]$$

and for each $a \in A$,

$$\varphi_i(a)_1 = a \land e_i, \quad \varphi_i(a)_2 = a \land e'_i.$$  

According to the isomorphism $\psi$ we infer that the mapping

$$\psi_i : X_{i1} \to L_i$$

defined by

$$\psi_i(x) = \psi(x)_i \quad \text{for each } x \in X_{i1}$$

is an isomorphism of $X_{i1}$ onto $L_i$.

Therefore in view of 6.2 we obtain

Lemma 6.3. There is a pseudo MV-algebra $B_i$ such that

(i) $L(B_i) = L_i$;

(ii) $\psi_i$ is an isomorphism of $X_i$ onto $B_i$.

Consider the pseudo MV-algebra

$$\prod_{i \in I} B_i = B.$$
Then in view of 6.3 we have \( L(\mathcal{B}) = \prod_{i \in I} L_i \).

Since all basic pseudo MV-operations in \( \mathcal{B} \) are calculated component-wise, from 6.3 and from the direct product decomposition \( \varphi_i \) we infer that the mapping \( \psi \) is a homomorphism of \( \mathcal{A} \) onto \( \mathcal{B} \).

From this and from the fact that \( \psi \) is a bijection we conclude that \( \psi \) is an isomorphism of \( \mathcal{A} \) onto \( \mathcal{B} \). In other words, we obtained a direct product decomposition

\[
\psi : \mathcal{A} \rightarrow \prod_{i \in I} \mathcal{B}_i
\]

which is induced by the direct product decomposition (3) of the lattice \( L \).

Summarizing, we have

**Theorem 6.4.** Let \( \mathcal{A} \) be a pseudo MV-algebra and \( L = L(\mathcal{A}) \). Then each direct product decomposition of \( L \) induces a direct product decomposition of \( \mathcal{A} \).

According to [7], any two direct product decompositions of a lattice have isomorphic refinements. From this and from 6.4 we conclude

**Theorem 6.5.** Any two direct product decompositions of a pseudo MV-algebra have isomorphic refinements.

**References**


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