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COMPLETE SPACELIKE HYPERSURFACES WITH CONSTANT SCALAR CURVATURE

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Abstract. In this paper, we characterize the n-dimensional ($n \geq 3$) complete spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures one of which is simple.We show that M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along $M_1^{n-1}(s)$ the principal curvature λ of multiplicity $n-1$ is constant and $M_1^{n-1}(s)$ is umbilical in S_1^{n+1} and is contained in an $(n-1)$-dimensional sphere $S^{n-1}\big(c(s)\big) = E^n(s) \cap S_1^{n+1}$ and is of constant curvature $\Big(\frac{d\{\log|\lambda^2-(1-R)|^{1/n}\}}{ds}\Big)^2 - \lambda^2 + 1$,where s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, $E^n(s)$ is an n-dimensional linear subspace in R_1^{n+2} which is parallel to a fixed $E^n(s_0)$ and $u = \Big|\lambda^2 - (1-R)\Big|^{-\frac{1}{n}}$ satisfies the ordinary differental equation of order 2, $\frac{d^2u}{ds^2} - u\Big(\pm\frac{n-2}{2}\frac{1}{u^n} + R - 2\Big) = 0$.

1. Introduction

Let R_1^{n+2} be the $(n+2)$-dimensional Lorentz-Minkowski space and S_1^{n+1} be the de Sitter space given by $S_1^{n+1} = \{p \in R_1^{n+2} \mid \langle p,p\rangle p = 1\}$. A hypersurface M^n of S_1^{n+1} is said to be spacelike if the induced metric on M^n from that of ambient space is positive definite. In [4] we investigated the spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures whose multiplicities are greater than 1. We showed that

Theorem 1.1 ([4]). *Let M^n be an n-dimensional complete spacelike hypersurface in S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than 1, then M^n is isometric to the Riemannian product $H^k(\sinh r) \times S^{n-k}(\cosh r)$, $1 < k < n-1$.*

As we know that Otsuki [3] characterized the minimal hypersurfaces in a Riemannian manifold $\bar{M}$ of constant curvature $\bar{c}$ with two distinct principal curvatures

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one of which is simple and Cheng [2] investigated the n-dimensional oriented complete hypersurfaces ($n \geq 3$) in Euclidean space R^{n+1} with constant scalar curvature and with two distinct principal curvatures one of which is simple. It is natural and important to investigate the spacelike hypersurfaces M^n in a de Sitter space S_1^{n+1} with constant scalar curvature and with two distinct principal curvatures one of which is simple. In this paper, we obtain the following

Theorem 1.2. *Let M^n be an n-dimensional ($n \geq 3$) complete spacelike hypersurface in a de Sitter space S_1^{n+1} with constant scalar curvature $n(n-1)R$ and with two distinct principal curvatures one of which is simple, then M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$, along $M_1^{n-1}(s)$ the principal curvature λ of multiplicity $n-1$ is constant and $M_1^{n-1}(s)$ is umbilical in S_1^{n+1} and is contained in an $(n-1)$-dimensional sphere $S^{n-1}\big(c(s)\big) = E^n(s) \cap S_1^{n+1}$ and is of constant curvature $\big(\frac{d\{\log|\lambda^2-(1-R)|^{1/n}\}}{ds}\big)^2 - \lambda^2 + 1$, where s is the arc length of an orthogonal trajectory of the family $M_1^{n-1}(s)$, $E^n(s)$ is an n-dimensional linear subspace in R_1^{n+2} which is parallel to a fixed $E^n(s_0)$ and $u = \big|\lambda^2 - (1-R)\big|^{-\frac{1}{n}}$ satisfies the ordinary differental equation of order 2*

$$\frac{d^2u}{ds^2} - u\Big(\pm\frac{n-2}{2}\frac{1}{u^n} + R - 2\Big) = 0\,.$$

2. Preliminaries

Let M^n be an n-dimensional spacelike hypersurfaces in S_1^{n+1}, we choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+1}$ in S_1^{n+1} such that at each point of $M^n, e_1, \dots, e_n$ span the tangent space of M^n and form an othonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1\,; \quad 1 \leq i, j, k, \dots \leq n\,.$$

Let $\omega_1, \dots, \omega_{n+1}$ be the dual frame field so that the semi-Riemannian metric of S_1^{n+1} is given by $d\bar{s}^2 = \sum_i \omega_i^2 - \omega_{n+1}^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_{n+1} = -1$.

The structure equations of S_1^{n+1} are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B\,, \quad \omega_{AB} + \omega_{BA} = 0\,, \tag{2.1}$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}\,, \tag{2.2}$$

where

$$\Omega_{AB} = -\frac{1}{2}\sum_{C,D} K_{ABCD}\,\omega_C \wedge \omega_D\,, \tag{2.3}$$

$$K_{ABCD} = \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})\,. \tag{2.4}$$

Restrict these forms to M^n, we have

$$\omega_{n+1} = 0\,. \tag{2.5}$$

Cartan's Lemma implies that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j\,, \quad h_{ij} = h_{ji}\,. \tag{2.6}$$

The structure equations of M^n are

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j\,, \quad \omega_{ij} + \omega_{ji} = 0\,, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l\,, \tag{2.8}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk})\,, \tag{2.9}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2.10}$$

is the second fundamental form of M^n.

From the above equation, we have

$$n(n-1)R = n(n-1) - n^2H^2 + |h|^2\,, \tag{2.11}$$

where $n(n-1)R$ is the scalar curvature of M^n, H is the mean curvature, and $|h|^2 = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form of M^n.

The Codazzi equation and the Ricci identity are

$$h_{ijk} = h_{ikj}\,, \tag{2.12}$$

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}\,, \tag{2.13}$$

where h_{ijk} and h_{ijkl} denote the first and the second covariant derivatives of h_{ij}.

We choose $e_1, \ldots, e_n$ such that $h_{ij} = \lambda_i\delta_{ij}$. From (2.6) we have

$$\omega_{n+1i} = \lambda_i\omega_i\,, \quad i = 1, 2, \ldots, n\,. \tag{2.14}$$

Hence, we have from the structure equations of M^n

$$\begin{aligned} d\omega_{n+1i} &= d\lambda_i \wedge \omega_i + \lambda_i d\omega_i \\ &= d\lambda_i \wedge \omega_i + \lambda_i \sum_j \omega_{ij} \wedge \omega_j. \end{aligned} \tag{2.15}$$

On the other hand, we have on the curvature forms of S_1^{n+1},

$$\begin{aligned} \Omega_{n+1i} &= -\frac{1}{2}\sum_{C,D} K_{n+1iCD}\omega_C \wedge \omega_D \\ &= \frac{1}{2}\sum_{C,D}(\delta_{n+1C}\delta_{iD} - \delta_{n+1D}\delta_{iC})\omega_C \wedge \omega_D \\ &= \omega_{n+1} \wedge \omega_i = 0\,. \end{aligned} \tag{2.16}$$

Therefore, from the structure equations of S_1^{n+1}, we have

$$\begin{aligned} d\omega_{n+1i} &= \sum_j \omega_{n+1j}\wedge\omega_{ji} - \omega_{n+1n+1}\wedge\omega_{n+1i} + \Omega_{n+1i} \\ &= \sum_j \lambda_j\omega_{ij}\wedge\omega_j . \end{aligned} \tag{2.17}$$

From (2.15) and (2.17), we obtain

$$d\lambda_i\wedge\omega_i + \sum_j(\lambda_i-\lambda_j)\omega_{ij}\wedge\omega_j = 0\,. \tag{2.18}$$

Putting

$$\psi_{ij} = (\lambda_i-\lambda_j)\omega_{ij}\,. \tag{2.19}$$

Then $\psi_{ij} = \psi_{ji}$. (2.18) can be written as

$$\sum_j(\psi_{ij} + \delta_{ij}d\lambda_j)\wedge\omega_j = 0\,. \tag{2.20}$$

By E. Cartan's Lemma, we get

$$\psi_{ij} + \delta_{ij}d\lambda_j = \sum_k Q_{ijk}\omega_k\,, \tag{2.21}$$

where Q_{ijk} are uniquely determined functions such that

$$Q_{ijk} = Q_{ikj}\,. \tag{2.22}$$

3. Proof of theorem

We firstly have the following Proposition 3.1 due to [1], which original due to Otsuki [3] for Riemannian space forms.

Proposition 3.1 ([1]). *Let M^n be a spacelike hypersurface in S_1^{n+1} such that the multiplicities of the principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature λ is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

Proof of Theorem 1.2. Let M^n be an n-dimensional complete spacelike hypersurface with constant scalar curvature and with two distinct principal curvatures one of which is simple, that is, without lose of generality, we may assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda\,,\quad \lambda_n = \mu\,,$$

where λ_i for $i = 1, 2, \ldots, n$ are the principal curvatures of M^n. Since the scalar curvature $n(n-1)R$ is constant, from (2.11), we obtain

$$n(n-1)(1-R) = (n-1)(n-2)\lambda^2 + 2(n-1)\lambda\mu\,. \tag{3.1}$$

If $\lambda = 0$ at some points, then $R = 1$ at these points from (3.1), since R is constant, we know $R = 1$ on M^n. Since these principal curvatures λ and μ are continuous on M^n, from (3.1) and $R = 1$ we obtain $\lambda = 0$ on M^n. Hence, from the Gauss equation, the sectional curvature of M^n $R_{ijij} = 1-\lambda\mu = 1 > 0$, by Myers' theorem we know that M^n is compact. From the result of Zheng [6, 5], we know that M^n is a totally umbilical spacelike hypersurface. This is impossible because we assumed that M^n is of two distinct principal curvatures. Hence, we can assume $\lambda \neq 0$ on M^n. From (3.1), we have

$$\mu = \frac{n(1-R)}{2\lambda} - \frac{(n-2)\lambda}{2}\,. \tag{3.2}$$

Therefore, we get

$$\lambda - \mu = n\frac{\lambda^2-(1-R)}{2\lambda} \neq 0\,,$$

we know $\lambda^2-(1-R)\neq 0$.

Let $u = \left|\lambda^2-(1-R)\right|^{-\frac{1}{n}}$. We denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\,\omega_k\,, \quad d\mu = \sum_{k=1}^{n} \mu_{,k}\,\omega_k\,. \tag{3.3}$$

From Proposition 3.1, we have

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x)\,. \tag{3.4}$$

From (3.2), we have

$$d\mu = \Big[-\frac{n(1-R)}{2\lambda^2} - \frac{n-2}{2}\Big]\,d\lambda\,. \tag{3.5}$$

Hence, we also have

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on} \quad M_1^{n-1}(x)\,. \tag{3.6}$$

In this case, we may consider locally λ is a function of the arc length s of the integral curve of the principal vector field e_n corresponding to the principal curvature μ. From (2.21) and (3.4), we have for $1 \leq j \leq n-1$,

$$\begin{aligned} d\lambda = d\lambda_j &= \sum_{k=1}^{n} Q_{jjk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{jjk}\omega_k + Q_{jjn}\omega_n = \lambda_{,n}\,\omega_n\,. \end{aligned} \tag{3.7}$$

Therefore, we have

$$Q_{jjk} = 0\,, \quad 1 \leq k \leq n-1\,, \quad \text{and} \quad Q_{jjn} = \lambda_{,n}\,. \tag{3.8}$$

By (2.21) and (3.6), we have

$$\begin{aligned} d\mu = d\lambda_n &= \sum_{k=1}^{n} Q_{nnk}\omega_k \\ &= \sum_{k=1}^{n-1} Q_{nnk}\omega_k + Q_{nnn}\omega_n = \sum_{i=1}^{n} \mu_{,i}\,\omega_i = \mu_{,n}\,\omega_n\,. \end{aligned} \tag{3.9}$$

Hence, we obtain

$$Q_{nnk} = 0\,, \quad 1 \le k \le n-1\,, \quad \text{and} \quad Q_{nnn} = \mu_{,n}\ . \tag{3.10}$$

From (3.5), we get

$$Q_{nnn} = \mu_{,n} = \Big[-\frac{n(1-R)}{2\lambda^2} - \frac{n-2}{2}\Big]\lambda_{,n}\ . \tag{3.11}$$

From the definition of ψ_{ij}, if $i \neq j$, we have $\psi_{ij} = 0$ for $1 \le i \le n-1$ and $1 \le j \le n-1$. Therefore, from (2.21), if $i \neq j$ and $1 \le i \le n-1$ and $1 \le j \le n-1$ we have

$$Q_{ijk} = 0\,, \quad \text{for any} \quad k\,. \tag{3.12}$$

By (2.21), (3.8), (3.10), (3.11) and (3.12), we get

$$\begin{aligned}\psi_{jn} &= \sum_{k=1}^{n} Q_{jnk}\omega_k \\ &= Q_{jjn}\omega_j + Q_{jnn}\omega_n = \lambda_{,n}\,\omega_j\ .\end{aligned} \tag{3.13}$$

Since λ and μ are two distinct principal curvatures of M^n, we have

$$\lambda - \mu = n\frac{\lambda^2 - (1-R)}{2\lambda} \neq 0\,.$$

From (2.19), (3.2)and (3.13)we have

$$\begin{aligned}\omega_{jn} &= \frac{\psi_{jn}}{\lambda-\mu} = \frac{\lambda_{,n}}{\lambda-\mu}\omega_j \\ &= \frac{\lambda_{,n}}{\lambda - [\frac{n(1-R)}{2\lambda} - \frac{n-2}{2}\lambda]}\omega_j \\ &= \frac{2\lambda\lambda_{,n}}{n[\lambda^2-(1-R)]}\,\omega_j\ .\end{aligned} \tag{3.14}$$

Therefore, from the structure equations of M^n we have

$$d\omega_n = \sum_{k=1}^{n-1}\omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0\,.$$

Therefore, we may put $\omega_n = ds$. By (3.7) and (3.9), we get

$$d\lambda = \lambda_{,n}\, ds\,, \quad \lambda_{,n} = \frac{d\lambda}{ds}\,,$$

and

$$d\mu = \mu_{,n}\, ds\,, \quad \mu_{,n} = \frac{d\mu}{ds}\ .$$

Then we have

$$\omega_{jn} = \frac{2\lambda\lambda_{,n}}{n[\lambda^2-(1-R)]}\,\omega_j = \frac{2\lambda\frac{d\lambda}{ds}}{n[\lambda^2-(1-R)]}\,\omega_j = \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\,\omega_j\,. \tag{3.15}$$

(3.15) shows that the integral submanifold $M_1^{n-1}(x)$ corresponding to λ and s is umbilical in M^n and S_1^{n+1}.

From (3.15) and the structure equations of S_1^{n+1}, we have

$$\begin{aligned} d\omega_{jn} &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn}\wedge\omega_{nn} + \omega_{jn+1}\wedge\omega_{n+1n} + \Omega_{jn} \\ &= \sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_{kn} + \omega_{jn+1}\wedge\omega_{n+1n} - \omega_j\wedge\omega_n \\ &= \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k - (\lambda\mu+1)\omega_j\wedge ds\,. \end{aligned}$$

From (3.15) we have

$$\begin{aligned} d\omega_{jn} &= \frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j + \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}d\omega_j \\ &= \frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2}ds\wedge\omega_j + \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n}\omega_{jk}\wedge\omega_k \\ &= \Big\{-\frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2} + \Big[\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big]^2\Big\}\omega_j\wedge ds \\ &\quad + \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\sum_{k=1}^{n-1}\omega_{jk}\wedge\omega_k\,. \end{aligned}$$

From the above two equalities, we have

$$\frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2} - \Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big\}^2 - (\lambda\mu+1) = 0\,. \tag{3.16}$$

From (3.2) we get

$$\frac{d^2\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds^2} - \Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big\}^2 + \frac{n-2}{2}\big[\lambda^2-(1-R)\big] + R - 2 = 0\,. \tag{3.17}$$

Since we define $u = \big|\lambda^2-(1-R)\big|^{-\frac{1}{n}}$, we obtain from the above equation

$$\frac{d^2u}{ds^2} - u\Big(\pm\frac{n-2}{2}\frac{1}{u^n} + R - 2\Big) = 0\,. \tag{3.18}$$

Since S_1^{n+1} is an $(n+1)$-dimensional de Sitter space of constant 1 in R_1^{n+2}. We consider the frame $e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}$ in R_1^{n+2}. Since the second fundamental form of S_1^{n+1} as the hypersurface R_1^{n+2} is given by $h_{AB} = -\sum_B \epsilon_B \delta_{AB}$, we have

$$\omega_{n+1n+2} = 0\,, \quad \text{and} \quad \omega_{in+2} = -\omega_i\,.$$

Then, from (2.14), (3.15) and (3.16), we have

$$\begin{aligned}
de_i &= \sum_{j=1}^{n-1} \omega_{ij} e_j + \omega_{in} e_n + \omega_{in+1} e_{n+1} + \omega_{in+2} e_{n+2} \\
&= \sum_{j=1}^{n-1} \omega_{ij} e_j + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\, \omega_i e_n - \lambda \omega_i e_{n+1} - e_{n+2}\omega_i \\
&= \sum_{j=1}^{n-1} \omega_{ij} e_j + \Big[\frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big]\, \omega_i\,.
\end{aligned}$$

On the other hand, by means of (3.16) we get

$$\begin{aligned}
& d\Big\{\frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big\} = d\Big\{\frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\Big\}\, e_n \\
&\quad + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\, de_n - d\lambda e_{n+1} - \lambda de_{n+1} - de_{n+2} \\
&= \Big\{\frac{d^2\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds^2} e_n - \frac{d\lambda}{ds} e_{n+1}\Big\}\, ds \\
&\quad + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} \Big(\sum_{j=1}^{n-1} \omega_{nj} e_j + \omega_{nn+1} e_{n+1} + \omega_{nn+2} e_{n+2}\Big) \\
&\quad - \lambda\Big(\sum_{j=1}^{n-1} \omega_{n+1j} e_j + \omega_{n+1n} e_n + \omega_{n+1n+2} e_{n+2}\Big) - \sum_{j=1}^{n-1} \omega_i e_j - \omega_n e_n \\
&= \Big\{\frac{d^2\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds^2} e_n - \frac{d\lambda}{ds} e_{n+1}\Big\}\, ds \\
&\quad + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds} \Big(\sum_{j=1}^{n-1} \omega_{nj} e_j - \mu \omega_n e_{n+1} - \omega_n e_{n+2}\Big) \\
&\quad - \lambda\Big(\lambda \sum_{j=1}^{n-1} \omega_j e_j + \mu \omega_n e_n\Big) - \sum_{j=1}^{n-1} \omega_i e_j - \omega_n e_n \\
&\equiv \Big[\frac{d^2\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds^2} - \lambda\mu - 1\Big]\, e_n \omega_n \\
&\quad - \Big\{\frac{d\lambda}{ds} + \frac{d\{\log|\lambda^2 - (1-R)|^{\frac{1}{n}}\}}{ds}\, \mu\Big\} e_{n+1} \omega_n
\end{aligned}$$

$$- \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds} e_{n+2}\omega_n \qquad (\mathrm{mod}\{e_1,\dots,e_{n-1}\})$$
$$= \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big\}\, ds\,.$$

We put

$$W = e_1 \wedge \cdots \wedge e_{n-1} \wedge \Big\{\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds} e_n - \lambda e_{n+1} - e_{n+2}\Big\}\,.$$

Therefore we have

$$dW = \frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds} W ds\,, \tag{3.19}$$

(3.19) shows that n-vector W in R_1^{n+2} is constant along $M_1^{n-1}(x)$. Hence, there exists an n-dimensional linear subspace $E^n(s)$ in R_1^{n+2} containing $M_1^{n-1}(x)$. By (3.19), the n-vector field W depends only on s and by integrating it, we get

$$W = \Big\{\frac{\lambda^2(s)-(1-R)}{\lambda^2(s_0)-(1-R)}\Big\}^{\frac{1}{n}} = W(s_0)\,.$$

Hence, we have that $E^n(s)$ is parallel to $E^n(s_0)$ in R_1^{n+2}.

Since $\Omega_{ij} = -\omega_i \wedge \omega_j$, from (2.2) the curvature of $M_1^{n-1}(x)$ is given by

$$d\omega_{ij} - \sum_{k=1}^{n-1} \omega_{ik} \wedge \omega_{kj} = \omega_{in} \wedge \omega_{nj} - \omega_{in+1} \wedge \omega_{n+1j} - \omega_i \wedge \omega_j$$
$$= -\Big\{\Big(\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\Big)^2 - \lambda^2 + 1\Big\}\omega_i \wedge \omega_j\,.$$

Therefore we know that the curvature of $M_1^{n-1}(x)$ is $\big(\frac{d\{\log|\lambda^2-(1-R)|^{\frac{1}{n}}\}}{ds}\big)^2 - \lambda^2 + 1$ and $M_1^{n-1}(x)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap S_1^{n+1}$ of the intersection of S_1^{n+1} and an n-dimensional linear subspace $E^n(s)$ in R_1^{n+2} which is parallel to a fixed $E^n(s_0)$. This completes the proof of Theorem 1.2.

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