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Computation of rigidity of order $\frac{n^2}{r}$ for one simple matrix

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Abstract. We shall compute the exact value of rigidity of the triangular matrix with entries 0 and 1.

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Classification: 15A03, 68Q15

Let F be an arbitrary field, let M be a square matrix of type n . The rigidity of M is the function depending on $r \in \{0, 1, \dots, n\}$, defined by

$$R_M^F(r) = \min\{|B|, M = A + B, r(A) \leq r\},$$

where $|B|$ denotes the number of nonzero elements in B and $r(A)$ denotes the rank of A . Intuitively, $R_M^F(r)$ is the minimal number of changes in M needed to reduce the rank to a value less or equal to r . The concept of rigidity was introduced J. Valiant [3] in connection with lower bounds to the size of circuits. He showed that a sufficiently large lower bound to the rigidity of a matrix implies that the transformation determined by the matrix cannot be computed by a linear size circuit. It is an open problem to find such matrices. So far only small lower bounds to the rigidity of explicitly given matrices have been proved. Razborov [2] proved an $\Omega(\frac{n^2}{r})$ lower bound to the rigidity of the matrix of the generalized Fourier transform and the inverse matrix of the Vandermonde matrix, Alon [1] proved an $\Omega(\frac{n^2}{r^2})$ for Hadamard matrices.

We shall determine the exact value of the rigidity of the triangular matrix

$$T_n = (t_{ij})_{i,j=1}^n, \quad t_{ij} = \begin{cases} 1, & i \geq j, \\ 0, & i < j. \end{cases}$$

Theorem 1. *Let $r < n$ be given and determine k and Δ by*

$$\begin{aligned} n &= k(2r + 1) + r + \Delta = \\ &= r(2k + 1) + k + \Delta, \\ (1) \quad k &\geq 0, 1 \leq \Delta \leq 2r + 1. \end{aligned}$$

Then

$$R_{T_n}^F(r) = \frac{(n - r + \Delta)(n + r - \Delta + 1)}{2(2r + 1)}.$$

Note that for n, r large but r small in comparison with n ,

$$R_{T_n}^F(r) \approx \frac{n^2}{4r}.$$

We shall say that

$$M = A + B$$

is a decomposition (of rank r , if $r(A) = r$), $|B|$ is the number of changes, $|b_i|$ is the number of changes in the i -th row, if b_i is the i -th row of B .

If $|B| = R_M^F(r)$, we shall say that the decomposition is optimal.

Speaking of linear dependence of rows of a decomposition we mean linear dependence of the rows of A .

The proof of Theorem 1 is based on the following lemma:

Lemma 1. *Let $r < n$ and let k be given by (1). Then in any decomposition of T_n of rank at most r there is a row containing at least $k + 1$ changes.*

We shall also determine the optimal decomposition of T_n .

Theorem 2. *All optimal decompositions of rank r of the matrix T_n have the form (2), (3) given in the proof of Theorem 1 below.*

The proof of Theorem 2 is based on the following lemmas:

Lemma 2. *Let n and $r < n$ be given, let k be determined by (1) and let an optimal decomposition of T_n be given. Then $k + 1$ is the maximum number of changes in a row.*

Lemma 3. *Let an optimal decomposition of T_n be given. Then deleting a row with the maximal number of changes and the corresponding column with the same index leads to an optimal decomposition of T_{n-1} .*

Proofs.

PROOF OF LEMMA 1: Let $T_n = A + B$ be a decomposition of rank r . Let t_j , resp. a_j, b_j be the j -th row of T_n , resp. A, B . Suppose for contradiction that the maximal number of changes in a row is k . Let us take $r + 1$ rows with indices belonging to the set

$$S = \{k + 1, k + 1 + 1(2k + 1), k + 1 + 2(2k + 1), \dots, k + 1 + r(2k + 1)\}.$$

These rows must be linearly dependent, i.e.

$$\sum_{j \in S'} \alpha_j a_j = 0$$

for some $0 \neq S' \subset S, |S'| = s' \leq r + 1, \alpha_j \neq 0$ for all $j \in S'$. Then

$$\sum_{S'} \alpha_j t_j = \sum_{S'} \alpha_j b_j$$

and, consequently,

$$|\sum_{S'} \alpha_j t_j| \leq \sum_{S'} |\alpha_j b_j| \leq s'k.$$

Denote $N = |\sum_{S'} \alpha_j t_j|$. The vector $\sum_{S'} \alpha_j t_j$ has the form

$$(c_1, \dots, c_1, c_2, \dots, c_2, \dots, c_{s'}, \dots, c_{s'}, 0, \dots, 0),$$

where the length of each constant section is at least $2k + 1$ except for the first section c_1, \dots, c_1 which can have the length $k + 1$. Observe that the last section $c_{s'}, \dots, c_{s'}$ cannot consist of zeros and that it is not possible that two consecutive sections consist of zeros. It follows that:

1. With exception of the case when the first section c_1, \dots, c_1 consists of nonzero elements and has length $k + 1$,

$$N \geq \frac{s'}{2}(2k + 1) > s'k,$$

which is a contradiction.

2. In the remaining case,

$$\begin{aligned} N &\geq k + 1 + \frac{s' - 1}{2}(2k + 1) = \\ &= \frac{2s'k + s' + 1}{2} > s'k \end{aligned}$$

and the same contradiction appears again. □

PROOF OF THEOREM 1: For $m = n, n - 1, \dots, r + 1$ let us proceed in the following way:

Having any decomposition $T_m = A + B$ of rank at most r , we find a row containing at least $k + 1$ changes ($m = k(2r + 1) + r + \Delta$) and reduce the matrices T_m, A, B by deleting this row and the corresponding column with the same index. Thus, we obtain a decomposition of T_{m-1} of rank at most r .

This procedure applied to any decomposition of T_n (of rank at most r) shows that the total number of changes is at least

$$\begin{aligned} &\Delta(k + 1) + (2r + 1)k + (2r + 1)(k - 1) + \dots + (2r + 1)1 = \\ &= (2r + 1)\frac{k(k + 1)}{2} + \Delta(k + 1) = \frac{(n - r + \Delta)(n + r - \Delta + 1)}{2(2r + 1)} \end{aligned}$$

(due to (1)).

Now consider the concrete decomposition

$$(2) \quad T_n = A + B$$

PROOF OF THEOREM 2: We shall proceed by induction on n :

1) $n = 1, r = 0$:

$$B = (+1).$$

$$n = 0(2r + 1) + r + 1.$$

B has $\Delta = 1$ nonzero triangle of dimension $k + 1 = 1, 2r + 1 - \Delta = 0$ nonzero triangles of dimension $k = 0$.

2) $n - 1 \rightarrow n$

Let n fulfil (1) and let the i -th row contain $k + 1$ changes. The induced decomposition of the $(n - 1)$ by $(n - 1)$ matrix which arises by deleting the i -th row and column, is an optimal decomposition (Lemma 3) and thus has the form (2), (3). Having this in mind, we can show schematically all the variants of the rows $i_0 - 1, i_0$ and $i_0 + 1$, with the changes in the $(i_0 - 1)$ st and $(i_0 + 1)$ st rows shown by the signs x :

$$\begin{array}{l}
 \begin{array}{cccccccccccc}
 1 & 1 & x & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 (i)(a) & 1 & 1 & \underline{1} & \underline{1} & \underline{1} & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & x & x & x & x & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \\
 \\
 \begin{array}{cccccccccccc}
 1 & 1 & 1 & 1 & 1 & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 (b) & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & x & 0 & 0 & 0 & 0 & 0
 \end{array} \\
 \\
 \begin{array}{cccccccccccc}
 1 & 1 & 1 & 1 & 1 & x & x & x & x & x & 0 & 0 & 0 \\
 (c) & 1 & 1 & 1 & 1 & 1 & 1 & \underline{0} & \underline{0} & \underline{0} & \underline{0} & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & x & x & x & 0 & 0 & 0
 \end{array} \\
 \\
 \begin{array}{cccccccccccc}
 1 & 1 & x & x & x & 0 & \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & 0 & 0 & 0 \\
 (ii)(a) & 1 & 1 & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \overline{0} & \overline{0} & \overline{0} & \overline{0} & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & x & x & x & 0 & 0 & 0
 \end{array} \\
 \\
 \begin{array}{cccccccccccc}
 1 & 1 & x & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 (b) & 1 & 1 & \underline{1} & \underline{1} & \underline{1} & \underline{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & x & x & x & x & 0 & 0 & 0
 \end{array} \\
 \\
 \begin{array}{cccccccccccc}
 1 & x & x & x & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 (c) & 1 & 1 & 1 & 1 & 1 & 1 & \underline{0} & \underline{0} & \underline{0} & \underline{0} & 0 & 0 & 0 \\
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & x & x & x & 0 & 0 & 0
 \end{array}
 \end{array}$$

It is evident that the changes of the i -th row shown by underline (resp. overline) are optimal. The situation $(i)(b)$ cannot occur since there are no changes in the i -th row needed. Neither $(i)(a), (c)$ can occur since the i -th row would not be a row with the maximal number of changes. Thus only the variants $(ii)(a), (b), (c)$ remain. The situation $(ii)(b)$ means the increase of an “(+1)-triangle”, $(ii)(c)$ means the increase of a “(-1)-triangle” and $(ii)(a)$ enables both of them.

In each case, the form (2), (3) of the decomposition of T_n is kept.

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