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TO THE DEFINITION OF THE GLOBAL TRANSFORMATION IN 2 – DIMENSIONAL LINEAR SPACES OF CONTINUOUS FUNCTIONS

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Dedicated to my father on his 65th birthday

In this article we shall be concerned with the question relating to the equivalence of definitions of the global transformation in two 2-dimensional linear spaces of continuous functions. We will apply the definitions of transformation introduced by O.Borůvka [1], F.Neuman [2] and K.Stach [3], in studying the spaces of solutions of the second-order linear differential equations in the Jacobian form, the spaces of solutions of the n-th order homogeneous linear differential equations, the 2-dimensional linear spaces of continuous functions, respectively.

1. The letter R means the field of real numbers. The set of continuous functions defined on an open interval j will be written as $C^{(0)}(j)$.

We recall at this point the concept of dependence and that of independence of continuous functions on the interval j and express a definition of the 2-dimensional space of continuous functions.

Definition 1.1. ([3]) Suppose $y_1, y_2 \in C^{(0)}(j)$. We say that the functions y_1, y_2 are dependent on j if there exist such numbers $k_1, k_2 \in R$, $k_1^2 + k_2^2 > 0$ that the identity

$$k_1 y_1(t) + k_2 y_2(t) \equiv 0$$

is valid on j.

If for any two numbers $k_1, k_2 \in R$, $k_1^2 + k_2^2 > 0$ and for every open interval $j_1 \subset j$

$$k_1 y_1(t) + k_2 y_2(t) \not\equiv 0$$

holds on $j_1 \subset j$, we say that the functions y_1, y_2 are independent on the interval j.

Definition 1.2. ([3]) Suppose $y_1, y_2 \in C^{(0)}(j)$ are independent functions on an open interval j, and $k_1, k_2 \in R$ be arbitrary numbers.

The set S of all functions $k_1 y_1 + k_2 y_2$ is called a 2-dimensional space of continuous functions. The interval j is called the definition interval of the space S and the ordered pair (y_1, y_2) is termed the basis of the space S. For brevity we speak hereafter of S as the space generated by the functions y_1, y_2 with the definition interval j.

Lemma 1.1. ([3]) Suppose S is a space generated by functions $y_1, y_2 \in C^{(0)}(j)$ with a definition interval j and $u, v \in S$ be independent functions on j. Then every function $z \in S$ may be expressed on the interval j as

$$z = c_1 u + c_2 v , \qquad (1.1)$$

where $c_1, c_2 \in R$ are convenient numbers.

Corollary. The space S generated by functions $y_1, y_2 \in C^{(0)}(j)$ with a definition interval j is a linear space of dimension 2 over the field R.

2. We now express three definitions of the global transformation of two 2-dimensional spaces of continuous functions and will show their equivalence.

Consider two spaces S_1 and S_2 of continuous functions. Let S_1 have a definition interval j and a basis (y_1, y_2), and let S_2 have a definition interval J and a basis (Y_1, Y_2).

Definition 2.1. The space S_2 is said to be globally transformed onto the space S_1 exactly if there exists to the vectors $\underline{y} = (y_1, y_2)^T$, $\underline{Y} = (Y_1, Y_2)^T$

a) a bijection $h: j \longrightarrow J$, $h \in C^{(0)}(j)$,

b) a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$,

c) a matrix $A = \| a_{ik} \|$, $i,k = 1,2$, $a_{ik} \in R$, $\det A \neq 0$, such that the equality

$$\underline{y}(t) = Af(t)\, \underline{Y}\left[h(t)\right] \qquad (2.1)$$

holds for every $t \in j$, where $(.,.)^T$ denotes a transposed vector to the vector $(.,.)$.

Convention. The mapping of the column vector $\underline{Y}$, $\underline{Y} = (Y_1, Y_2)^T$ on the column vector $\underline{y}$, $\underline{y} = (y_1, y_2)^T$ defined by equation (2.1) will be denoted by $\mathcal{T}$ and written as

$$\mathcal{T}\underline{Y} = \underline{y}, \ \mathcal{T} = \langle Af,\ h \rangle .$$

The mapping $\mathcal{T}$ will be called the global transformation of the space S_2 onto the space S_1.

Definition 2.2. Let f, h be such functions that

a) is a bijection : $j \longrightarrow J$, $h \in C^{(0)}(j)$

b) $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$.

The space S_2 is said to be globally transformed onto the space S_1 if the functions

$$y(t) = f(t)\, Y\left[h(t)\right] \qquad (2.2)$$

are exactly all elements of the space S_1 for $Y \in S_2$.

Definition 2.3. The space S_2 is said to be globally transformed onto the space S_1 exactly if there exist

a) a bijection $h: j \to J$, $h \in C^{(0)}(j)$,

b) a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$

such that

1) there exists such a function $y \in S_1$ to every function $Y \in S_2$ that

$$y(t) = f(t)\, Y\left[h(t)\right] \tag{2.3}$$

holds for every $t \in j$.

2) there exists a function $Y \in S_2$ to every function $y \in S_1$ such that (2.3) is valid for every $t \in j$.

Remark. It should be noted here that definition 2.1. corresponds to the definition used by F.Neuman in [2] in case of the spaces of solutions of the homogeneous n-th order linear differential equations. Definition 2.2 is in the main used by O. Borůvka in [1] to define the global transformation of the linear second-order differential equations in the Jacobian form. Definition 2.3. is used by K.Stach in [3] in case of 2-dimensional spaces of continuous functions.

Lemma 2.1. Suppose the space S_2 is globally transformed onto the space S_1 by way of formula (2.1) following Definition 2.1. Then equality (2.1) is equivalent to equality

$$\underline{y}(t) = f(t)\, \underline{\tilde{Y}}\left[h(t)\right] \tag{2.4}$$

and $(\tilde{Y}_1, \tilde{Y}_2)$ is a basis of the space S_2, whereby

$$\begin{aligned} \tilde{Y}_1 &= a_{11} Y_1 + a_{12} Y_2, \\ \tilde{Y}_2 &= a_{21} Y_1 + a_{22} Y_2 . \end{aligned} \tag{2.5}$$

P r o o f. Denoting $\underline{\tilde{Y}} = A\underline{Y}$ we observe that equalities

(2.1) and (2.4) are clearly equivalent and it follows from the assumption det A $\neq$ 0 that $(\tilde{Y}_1, \tilde{Y}_2)$ is a basis of the space S_2.

Lemma 2.2. *Definitions 2.1 and 2.2 of the global transformation of the space S_2 onto the space S_1 are equivalent.*

P r o o f. Suppose the space S_2 is globally transformed onto the space S_1 by Definition 2.1. Then there exist

a) a bijection $h: j \in J$, $h \in C^{(0)}(j)$,

b) a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$,

c) a matrix $A = \| a_{ik} \|$, $i,k = 1,2$, $a_{ik} \in R$, $\det A \neq 0$

such that the equality

$$\underline{y}(t) = A\, f(t)\, \underline{Y}\left[h(t)\right]$$

holds for every $t \in j$, where $\underline{Y} = (Y_1, Y_2)^T$, $\underline{y} = (y_1, y_2)^T$. Multiplying the vector equality of (2.1) from the left by the vector $\underline{K} = (k_1, k_2)$, $k_1 k_2 \in R$, yields

$$\underline{K}\,\underline{y}\,(t) = \underline{K}\, A\, f(t)\, \underline{Y}\left[h(t)\right] . \tag{2.6}$$

Setting $\tilde{\underline{Y}} = A\underline{Y}$, then (2.6) may be written as

$$\underline{K}\,\underline{y}\,(t) = \underline{K}\, f(t)\, \tilde{\underline{Y}}\left[h(t)\right]. \tag{2.7}$$

Hence we see that the element $Z = \underline{K}\,\tilde{\underline{Y}} \in S_2$ is mapped by (2.7) onto the element $z = \underline{K}\,\underline{y} \in S_1$.

It becomes readily apparent now that every element $z \in S_1$ has its pattern in S_2 because there exists exactly one vector $\underline{K}$ to the basis $\underline{y} = (y_1, y_2)$ and to the element $z \in S_1$ such that $z = \underline{K}\,\underline{y}$. Clearly, its pattern is the element $Z = \underline{K}\,\tilde{\underline{Y}} \in S_2$.

Thus, the space S_2 is globally transformed onto the space S_1 by Definition 2.2.

Suppose conversely that the space S_2 is globally transformed onto the space S_1 by Definition 2.2. If follows from Definition 2.2. that there exist functions f, h such that

a) h is a bijection : $j \rightarrow J$, $h \in C^{(0)}(j)$,

b) the function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$ and the element $Z \in S_2$ is mapped onto the element $z \in S_1$ given by equation (2.2). Hence we have

$$f[Y_1(h)] = u_1 , \quad f[Y_2(h)] = u_2$$

for the elements of the basis $Y_1, Y_2 \in S_2$, whereby $u_1, u_2 \in S_1$ are independent functions. Thus, there exist such numbers $c_{ik} \in R$, $i,k = 1,2$ that

$$u_1 = c_{11}\, y_1 + c_{12}\, y_2 ,$$

$$u_2 = c_{21}\, y_1 + c_{22}\, y_2 ,$$

whereby $\| c_{ik} \| \neq 0$. So, we have

$$c_{11}\, y_1 + c_{12}\, y_2 = f(t)\, Y_1 [h(t)] ,$$

$$c_{21}\, y_1 + c_{22}\, y_2 = f(t)\, Y_2 [h(t)] ,$$

whence

$$\underline{y}(t) = A\, f(t)\, \underline{Y} [h(t)] ,$$

where $A = C^{-1}$, $C = \| c_{ik} \|$, $i,k = 1,2$. Thus, S_2 is globally transformed into S_1 by Definition 2.1.

Lemma 2.3. *Definitions 2.1 and 2.3 of the global transformation of the space S_2 into the space S_1 are equivalent.*

P r o o f. Let the space S_2 be globally transformed on the space S_1 by Definition 2.1. We know from the above that equalit (2.1) is equivalent to equality (2.4), whereby (2.5) holds.

1) $Y \in S_2$ be an arbitrary element. We will show that there exists a function $y \in S_1$ such that (2.3) is true. Indeed, there exist numbers $\lambda_1, \lambda_2 \in R$ to the basis $(\tilde{Y}_1, \tilde{Y}_2)$ (discussed in Lemma 2.1) such that

$$Y = \lambda_1 \tilde{Y}_1 + \lambda_2 \tilde{Y}_2 .$$

On multiplying (2.4) by the vector (λ_1, λ_2), we obtain

$$(\lambda_1, \lambda_2)\ \underline{y}(t) = (\lambda_1, \lambda_2)\ f(t)\ \underline{\tilde{Y}}\ [h(t)]$$

i.e.

$$\lambda_1\ y_1 + \lambda_2\ y_2 = f(t)\ [\lambda_1 \tilde{Y}_1(h) + \lambda_2 \tilde{Y}_2(h)] = f(t)\ Y(h).$$

If we put $\lambda_1 y_1 + \lambda_2 y_2 = y$, we obtain (2.3).

2) Let $y \in S_1$ be an arbitrary element. We will show that there exist an $Y \in S_2$ so that (2.3) holds. Indeed, there exist numbers $\mu_1, \mu_2 \in R$ to the basis (y_1, y_2) so that

$$y = \mu_1\ y_1 + \mu_2\ y_2\ .$$

On multiplying (2.4) be the vector (μ_1, μ_2) we obtain

$$(\mu_1, \mu_2)\ \underline{y}(t) = (\mu_1, \mu_2)\ f(t)\ \underline{\tilde{Y}}\ [h(t)]$$

i.e.

$$y = \mu_1\ y_1 + \mu_2\ y_2 = f(t) [\mu_1\ \tilde{Y}_1\ [h(t)] + \mu_2\ \tilde{Y}_2\ [h(t)]]\ .$$

If we put $Y = \mu_1 \tilde{Y}_1 + \mu_2 \tilde{Y}_2$, we obtain (2.3).

Suppose conversely that the space S_2 is globally transformed onto the space S_1 by Definition 2.3. $(\tilde{Y}_1, \tilde{Y}_2)$ be a basis of the space S_2. From Definition 2.3 follows then the existence of

a) a bijection $h: j \to J$, $h \in C^{(0)}(j)$,

b) a function $f \in C^{(0)}(j)$, $f(t) \neq 0$ for $t \in j$

such that there exist functions $y_1 \in S_1$ and $y_2 \in S_1$ to functions $\tilde{Y}_1 \in S_2$ and $\tilde{Y}_2 \in S_2$ that in view of (2.3) the equalities

$$y_1(t) = f(t)\ \tilde{Y}_1\ [h(t)],\quad y_2(t) = f(t)\ \tilde{Y}_2\ [h(t)]$$

hold, respectively. By setting $\underline{y} = (y_1, y_2)^T$, $\underline{\tilde{Y}} = (\tilde{Y}_1, \tilde{Y}_2)^T$, we obtain

$$\underline{y}(t) = f(t)\,\tilde{Y}\left[h(t)\right] .$$

By Lemma 2.1 the last equality is equivalent to equality (2.1). Consequently, the space S_2 is globally transformed onto the space S_1 by Definition 2.1.

Theorem 2.1. Definitions 2.1, 2.2 and 2.3 of the global transformation of the space S_2 onto the space S_1 are equivalent.

P r o o f. The assertion follows from Lemmas 2.2. and 2.3.

3. The point mapping of the space S_2 onto the space S_1 defined by equation (2.7) and intermediated by the vector $\underline{K} = (k_1, k_2)$, $k_1, k_2 \in R$ will be denoted by $\mathcal{T}^*$ and written as

$$\mathcal{T}^*(\underline{K}\,\tilde{\underline{Y}}) = \underline{K}\,\underline{y}$$

or also as

$$\mathcal{T}^*(k_1 \tilde{Y}_1 + k_2 \tilde{Y}_2) = k_1 y_1 + k_2 y_2 . \tag{3.1}$$

We say that the mapping $\mathcal{T}$ which maps the basis (Y_1, Y_2) of the space S_2 on the basis (y_1, y_2) of the space S_1, induces the mapping $\mathcal{T}^*: S_2 \to S_1$ which with the aid of equation (2.7) maps the element $Z = \underline{K}\,\tilde{\underline{Y}}$ of the space S_2 onto the element $z = \underline{K}\,\underline{y}$ of the space S_1.

We shall now show that the mapping $\mathcal{T}^*$ of the space S_2 onto the space S_1 is a linear mapping and the global transformation $\mathcal{T} = \langle Af, h \rangle$ is a relation of equivalence on the set of the 2-dimensional spaces of the continuous functions. This will be discussed in the theorems below.

Theorem 3.1. The mapping $\mathcal{T}^*: S_2 \to S_1$ is a linear mapping.

P r o o f. Let $U_1, U_2 \in S_2$ be arbitrary elements and $k_1, k_2 \in R$ be arbitrary numbers. Since U_1, $U_2 \in S_2$, there exist numbers $\varkappa_{ik}$, $i,k = 1,2$ such that

$$U_1 = \mathcal{H}_{11}\tilde{Y}_1 + \mathcal{H}_{12}\tilde{Y}_2$$
$$U_2 = \mathcal{H}_{21}\tilde{Y}_1 + \mathcal{H}_{22}\tilde{Y}_2$$

for $\tilde{Y}_1, \tilde{Y}_2$ is a basis of the space S_2 with respect to the fact that the determinant $|a_{ik}| \neq 0$. On account of (3.1) we obtain

$$\mathcal{T}^* U_1 = \mathcal{T}^*(\mathcal{H}_{11}\tilde{Y}_1 + \mathcal{H}_{12}\tilde{Y}_2) = \mathcal{H}_{11} y_1 + \mathcal{H}_{12} y_2 ,$$
$$\mathcal{T}^* U_2 = \mathcal{T}^*(\mathcal{H}_{21}\tilde{Y}_1 + \mathcal{H}_{22}\tilde{Y}_2) = \mathcal{H}_{21} y_1 + \mathcal{H}_{22} y_2 ,$$

whence it follows

$$\mathcal{T}^*(k_1U_1+k_2U_2) = \mathcal{T}^*(k_1\mathcal{H}_{11}+k_2\mathcal{H}_{21})\tilde{Y}_1+(k_1\mathcal{H}_{12}+k_2\mathcal{H}_{22})\tilde{Y}_2 =$$
$$= (k_1\mathcal{H}_{11}+k_2\mathcal{H}_{21})y_1+(k_1\mathcal{H}_{12}+k_2\mathcal{H}_{22})y_2 =$$
$$= k_1(\mathcal{H}_{12}y_1+\mathcal{H}_{12}y_2)+k_2(\mathcal{H}_{21}y_1+\mathcal{H}_{22}y_2) =$$
$$= k_1(\mathcal{T}^* U_1) + k_2(\mathcal{T}^* U_2) .$$

Thus

$$\mathcal{T}^*(k_1U_1+k_2U_2) = k_1(\mathcal{T}^*U_1)+k_2(\mathcal{T}^*U_2) .$$

The mapping $\mathcal{T}^*$ is linear.

Theorem 3.2. *The global transformation $\mathcal{T} = \langle Af, h \rangle$ is a relation of the equivalence on the set of the 2-dimensional spaces of continuous functions.*

P r o o f. We have to show that the global transformation is reflexive, symmetric and transitive.

1) The space S is globally transformed by the global transformation $\mathcal{T} = \langle E, t \rangle$, where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, hence the relation of the global transformation is reflexive.

2) Let S_1 be the space of continuous functions with a definition interval j_1 and S_2 be the space of continuous

functions with a definition interval j_2. If the space S_2 is globally transformed on the space S_1, then the space S_1 is globally transformed on the space S_2.

Indeed, let $\mathcal{T} = \langle Af, h \rangle$, $\underline{y} = \mathcal{T}\underline{Y}$, be a global transformation of S_2 on S_1. Then

a) $h: j_1 \longrightarrow j_2$ is a bijection, $h \in C^{(0)}(j_1)$,

b) $f \in C^{(0)}(j_1)$, $f(t) \neq 0$ for $h \in j_1$,

c) $\det A \neq 0$, where $A = \| a_{ik} \|$, $i,k = 1,2$.

Since

a) $h^{-1}: j_2 \longrightarrow j_1$ is a bijection, $h^{-1} \in C^{(0)}(j_2)$

b) $F := \dfrac{1}{f(h^{-1})} \in C^{(0)}(j_2)$, $F(t) \neq 0$ for $t \in j_2$

c) $\det A^{-1} \neq 0$,

$\underline{Y} = \mathcal{T}^{-1}\underline{y}$, where $\mathcal{T}^{-1} = \langle A^{-1}F, h^{-1} \rangle$ is a global transformation of the space S_1 onto the space S_2. Hence the relation of the global transformation is symmetric.

3) Let S_i be the spaces of continuous functions with the definition intervals j_i, $i = 1,2,3$. If the space S_3 is globally transformed onto the space S_2 and the space S_2 is globally transformed onto the space S_1, then the space S_3 is globally transformed onto the space S_1.

Indeed, if

$$\mathcal{T}_1 = \langle Af, h \rangle , \quad \underline{V} = \mathcal{T}_1 \underline{Z}$$

is a global transformation of S_3 onto S_2, then

a) $h : j_2 \longrightarrow j_3$ is a bijection, $h \in C^{(0)}(j_2)$

b) $f \in C^{(0)}(j_2)$, $f(t) \neq 0$ for $t \in j_2$

c) $\det A \neq 0$, where $A = \| a_{ik} \|$, $i,k = 1,2$,

and if

$$\tau_2 = \langle Bg, l \rangle \;, \quad \underline{U} = \tau_2 \underline{V}$$

is a global transformation of S_2 onto S_1, then

a) $l : j_1 \to j_2$ is a bijection, $l \in C^{(0)}(j_1)$

b) $g \in C^{(0)}(j_1)$, $g(t) \neq 0$ for $t \in j_1$

c) $\det B \neq 0$, where $B = \| b_{ik} \|$, $i,k = 1,2$,

and it holds:

$$\tau_1 \underline{Z} = \underline{V} = Af\, \underline{Z}(h)$$

$$\tau_2 \underline{V} = \underline{U} = Bg\, \underline{V}(l) = Bg\, Af(l)\, Z[h(l)]$$

i.e.

$$\underline{U} = BA\, gf(l)\, \underline{Z}\left[h(l)\right] . \tag{3.3}$$

Since

a) $h(l) : j_1 \to j_3$ is a bijection, $h(l) \in C^{(0)}(j_1)$

b) $gf(l) \in C^{(0)}(j_1)$, $gf(l) \neq 0$ for $t \in j_1$

c) $\det BA \neq 0$,

it appears that equation (3.3) defines the global transformation of the space S_3 onto the space S_1. Hence, the relation of the global transformation is transitive.

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SOUHRN

K definici globální transformace lineárních prostorů spojitých funkcí dimenze 2

J i t k a L a i t o ch o v á

V článku je ukázána ekvivalence tří definic globální transformace lineárních prostorů spojitých funkcí dimenze 2, které používají O.Borůvka [1] v případě lineárních diferenciálních rovnic druhého řádu Jacobiho tvaru, F.Neuman [2] v případě prostorů řešení lineárních diferenciálních rovnic n-tého řádu bez pravé strany a K.Stach [3] v případě lineárních prostorů spojitých funkcí dimenze 2.

Je dokázáno, že globální transformace je relace ekvivalence na množině prostorů spojitých funkcí dimenze 2.

РЕЗЮМЕ

Заметка о глобальной трансформации двухразмерных линейных пространств непрерывных функций

Йитка Лаитохова

В работе доказана эквивалентность трех определений глобальной трансформации двухразмерных линейных пространств непрерывных функций, которыми пользуются О. Борувка [1] в случае линейных дифференциальных уравнений вида Якоби, Ф. Нейман [2] в случае пространств решений линейных дифференциальных уравнений н-го порядка без правой части и К. Стах [3]

в случае двухразмерных линейных пространств непрерывных функций.

Доказано, что глобальная трансформация представляет собой отношение эквивалентности на множестве двухразмерных линейных пространств непрерывных функций.

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