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## New properties of the concentric circle space and its applications to cardinal inequalities

SHU-HAO SUN, KOO-GUAN CHOO

*Abstract.* It is well-known that the concentric circle space has no  $G_\delta$ -diagonal nor any countable point-separating open cover. In this paper, we reveal two new properties of the concentric circle space, which are the weak versions of  $G_\delta$ -diagonal and countable point-separating open cover. Then we introduce two new cardinal functions and sharpen some known cardinal inequalities.

*Keywords:* concentric circle space, weak  $G_\delta$ -diagonal, point-separating \*-open cover, cardinal function

*Classification:* 54G05, 54A25

### 1. Concentric circle space and its new properties.

Let us first recall the definition of the concentric circle space or the Alexandroff double circle space. Let

$$C_i = \{(x, y) \mid x^2 + y^2 = i\}, \quad (i = 1, 2),$$

and let  $P : C_1 \rightarrow C_2$  be the projection of  $C_1$  onto  $C_2$  from the origin  $(0, 0)$ . Let  $X = C_1 \cup C_2$  and we define the neighbourhood system  $\{\mathcal{B}(z)\}$  of  $X$  as follows: let

$$\{\mathcal{B}(z)\} = \begin{cases} \{\{z\}\}, & \text{for } z \in C_2, \\ \{U_j(z)\}_{j=1}^\infty & \text{for } z \in C_1, \end{cases}$$

where

$$U_j(z) = V_j(z) \cup P(V_j(z) - \{z\}),$$

and  $V_j(z)$  is the arc of  $C_1$  with center at  $z$  and length  $1/j$ . Then such  $X$  (with the defined neighbourhood system) is called the *concentric circle space* or *Alexandroff double circle space*. It is well-known that the concentric circle space  $X$  is a compact  $T_2$  space (in fact,  $T_5$  space) (cf. [2]).

Next, we recall that a topological space  $Y$  has a  $G_\delta$ -diagonal, iff there exists a sequence of open covers  $\{\mathcal{U}_n\}$  of  $Y$  with

$$\bigcap_n \text{St}(y, \mathcal{U}_n) = \{y\}$$

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for each  $y \in Y$ , where

$$\text{St}(y, \mathcal{U}) = \bigcap \{B \in \mathcal{U} \mid y \in B\}.$$

A cover  $\mathcal{U}$  of  $Y$  is called *point-separating*, if for each  $y \in Y$ ,

$$\bigcap \{U \in \mathcal{U} \mid y \in U\} = \{y\}.$$

It is also well-known that the concentric circle space  $X$  is not metrizable, and so it has no  $G_\delta$ -diagonal nor any countable point-separating open cover. Although  $X$  has no  $G_\delta$ -diagonal, we will show that it has a weak  $G_\delta$ -diagonal as defined below. We will also show that  $X$  has a countable point-separating  $*$ -open cover as defined below.

**Definition.** Let  $Y$  be any topological space. Then a collection  $\mathcal{U}$  of subsets of  $Y$  is called a  *$*$ -open collection*, if for each  $y \in Y$ ,  $\text{St}(y, \mathcal{U})$  is an open set. Moreover, if for each  $y \in Y$ ,  $\text{St}(y, \mathcal{U})$  is a non-empty open set, then  $\mathcal{U}$  is called a  *$*$ -open cover*.

A space  $Y$  is said to have a *weak  $G_\delta$ -diagonal*, if there is a sequence  $\{\mathcal{U}_n\}$  of  $*$ -open covers such that

$$\bigcap_n \text{St}(y, \mathcal{U}_n) = \{y\},$$

for each  $y \in Y$ .

**Remark.** A collection of open sets is clearly a  $*$ -open collection. But the converse is not true. For example

$$\mathcal{U} = \{\{y\}\} \cup \{Y\}$$

is a  $*$ -open cover of  $Y$ , but it is not an open cover, if  $Y$  is not discrete. On the other hand, if for each  $\mathcal{V} \subseteq \mathcal{U}$ ,  $\mathcal{V}$  is a  $*$ -open collection, then it is easy to check that  $\mathcal{U}$  has to be an open collection.

**Lemma 1.** *A topological space  $Y$  has a weak  $G_\delta$ -diagonal, if there is a mapping  $g : Y \times \mathbb{N} \rightarrow \tau$ , where  $\tau$  is the topology of  $Y$ , such that for each  $y \in Y$ ,*

$$\bigcap_{n \in \mathbb{N}} g(y, n) = \{y\},$$

and for each  $n \in \mathbb{N}, x, y, \in Y, y \in g(x, n)$  implies  $x \in g(y, n)$ .

**PROOF:** Suppose that  $Y$  has a weak  $G_\delta$ -diagonal; i.e., suppose that  $Y$  has a sequence  $\{\mathcal{U}_n\}_{n=1}^\infty$  of  $*$ -open covers such that  $\bigcap_n \text{St}(y, \mathcal{U}_n) = \{y\}$  for each  $y \in Y$ . Define  $g : Y \times \mathbb{N} \rightarrow \tau$  by

$$g(y, n) = \text{St}(y, \mathcal{U}_n).$$

Then clearly  $g$  has the required properties.

Conversely, suppose that the mapping  $g$  with the required property is given. For each  $y \in Y$  and  $n \in \mathbb{N}$ , let

$$R_n(y) = \{\{y, x\} \mid x \in g(y, n)\}$$

and

$$\mathcal{U}_n = \bigcup_{y \in Y} R_n(y).$$

Then  $\{\mathcal{U}_n\}_{n=1}^\infty$  is the required sequence of  $*$ -open covers such that

$$\bigcap_n \text{St}(y, \mathcal{U}_n) = \{y\}$$

for each  $y \in Y$ . Firstly, for each  $n \in \mathbb{N}$ ,  $\mathcal{U}_n$  is a cover of  $Y$ . Next, for each  $y \in Y$  and  $n \in \mathbb{N}$ ,  $\text{St}(y, \mathcal{U}_n) = g(y, n)$ . Clearly  $g(y, n) \subseteq \text{St}(y, \mathcal{U}_n)$ . Now, if  $x \in \text{St}(y, \mathcal{U}_n)$ , then

$$y \in \bigcup_{x \in Y} R_n(x),$$

i.e.,  $y \in g(x, n)$  so that  $x \in g(y, n)$ . Thus  $\text{St}(y, \mathcal{U}_n) \subseteq g(y, n)$ . This completes the proof.  $\square$

**Proposition 1.** *The concentric circle space  $X$  has a weak  $G_\delta$ -diagonal.*

PROOF: Define  $g : X \times \mathbb{N} \rightarrow \tau$  by

$$g(x, n) = \begin{cases} U_n(x), & \text{if } x = z \in C_1, \\ (U_n(z) - \{z\}) \cup \{x\}, & \text{if } x = P(z) \in C_2, z \in C_1. \end{cases}$$

Then clearly for each  $n \in \mathbb{N}$ ,

$$\bigcup_{x \in X} g(x, n) = X,$$

for each  $x \in X$ ,

$$\bigcap_{n \in \mathbb{N}} g(x, n) = \{x\},$$

and for each  $x \in X, n \in \mathbb{N}, g(x, n)$  is open.

By Lemma 1, it remains to show that for each  $n \in \mathbb{N}$ , and for any  $x, y \in X$ ,  $x \in g(y, n)$  implies  $y \in g(x, n)$ . We divide this into four cases.

(i) Both  $x, y \in C_1$ . If

$$y \in g(x, n) = U_n(x) = V_n(x) \cup P(V_n(x) - \{x\}),$$

then  $y \in V_n(x)$  so that  $x \in V_n(y) \subseteq U_n(y) = g(y, n)$ .

(ii)  $x \in C_1$  and  $y \in C_2$ . Let  $y = P(z)$ , where  $z \in C_1$ . If

$$y \in g(x, n) = V_n(x) \cup P(V_n(x) - \{x\}),$$

then  $y \in P(V_n(x) - \{x\})$  so that  $z \in V_n(x) - \{x\}$  and thus

$$x \in V_n(z) - \{z\} \subseteq (U_n(z) - \{z\}) \cup \{y\} = g(y, n).$$

(iii)  $x \in C_2$  and  $y \in C_1$ . Let  $x = P(w)$ , where  $w \in C_1$ . If

$$y \in g(x, n) = (U_n(w) - \{w\}) \cup \{x\},$$

then  $y \in V_n(w) - \{w\}$  so that  $w \in V_n(y) - \{y\}$  and hence

$$x = P(w) \in P(V_n(y) - \{y\}) \subseteq U_n(y) = g(y, n).$$

(iv) Both  $x, y \in C_2$ . Let  $x = P(w)$  and  $y = P(z)$ , where  $w, z \in C_1$ . If

$$y \in g(x, n) = (U_n(a) - \{a\}) \cup \{x\},$$

then  $y \in P(V_n(w))$  so that  $z \in V_n(w)$ . Thus  $w \in V_n(z)$  and therefore

$$x = P(w) \in P(V_n(z)) \subseteq g(y, n).$$

This completes the proof. □

For convenience, we now modify slightly the basic sets in the Alexandroff double circle space as follows: let

$$X_i = [0, 1] \times \{i\}$$

replace  $C_i$  for  $i = 1, 2$ , and transform the projection  $P$  onto a mapping which maps  $(a, 1)$  into  $(a, 2)$  for each  $a \in [0, 1]$ . Since a circle is obtained by identifying the end points of  $[0, 1]$ , this is consistent with the previous definition.

The following proposition shows that although the Alexandroff double circle space  $X$  does not have any countable point-separating open cover, it does have a pointwise countable point-separating  $*$ -open cover.

**Proposition 2.** *For the Alexandroff double circle space  $X$ , there is a cover  $\mathcal{U}$  such that*

$$\bigcap \mathcal{U}_x = \bigcap \{B \in \mathcal{U} \mid x \in B\} = \{x\},$$

$|\mathcal{U}_x| \leq \omega_0$  and  $\mathcal{V}_x = \mathcal{U} \setminus \mathcal{U}_x$  is a  $*$ -open collection, for each  $x \in X$ , where  $|\mathcal{U}_x|$  denotes the cardinality of  $\mathcal{U}_x$  and  $\omega_0$  is the least infinite cardinality.

PROOF: Let  $\mathcal{Q}_i$  be the family of all non-empty open intervals with rational end points in  $X_i$ , for  $i = 1, 2$ . Then let  $\mathcal{U}$  be the collection

$$\mathcal{U} = \{\{x\}\}_{x \in X} \cup \{Q_1 \cup Q_2 \mid Q_1 \in \mathcal{Q}_1, Q_2 \in \mathcal{Q}_2\}.$$

Then  $\mathcal{U}$  is the cover having the desired properties.

Clearly,  $\mathcal{U}$  is a point-separating cover of  $X$  and we have

$$\bigcap \mathcal{U}_x = \bigcap \{B \in \mathcal{U} \mid x \in B\} = \{x\},$$

and  $|\mathcal{U}_x| \leq \omega_0$ , for each  $x \in X$  (i.e.,  $\mathcal{U}$  is a pointwise countable cover).

We now show that for each  $x \in X$ ,  $\mathcal{V}_x = \mathcal{U} \setminus \mathcal{U}_x$  is a \*-open collection. It suffices to show that for each  $B \in \mathcal{V}_x$ , if  $w \in B$ , then there exists a sequence  $\{B_j\} \subseteq \mathcal{V}_x$  such that  $w \in B_j$ , for each  $j$ , and  $\bigcup_j B_j$  is open. Clearly, we can take

$$B = Q_1 \cup Q_2, \quad (Q_i \in \mathcal{Q}_i, i = 1, 2).$$

Let  $w \in B = Q_1 \cup Q_2 \in \mathcal{V}_x$ . Then there are two cases:

(i)  $w \in Q_1$  and  $x \in P(Q_1)$ . Let  $x = (a, 2)$ , where  $a \in [0, 1]$ . Let  $\ell_1$  (resp.  $r_1$ ) denote the left (resp. right) end point of  $Q_1$ . Then there is an increasing sequence  $\{\ell_n\}$  of rational numbers and a decreasing sequence  $\{r_n\}$  such that  $\sup\{\ell_n\} = a$  and  $\inf\{r_n\} = a$ . Now let

$$D_j = (\ell_1, \ell_j) \times \{2\}, \quad E_j = (r_j, r_1) \times \{2\}, \quad (j = 1, 2, \dots).$$

Then  $D_j \cup Q_1$  and  $E_j \cup Q_1$  are in  $\mathcal{V}_x$ , for  $j = 2, 3, \dots$ , and

$$\bigcup_{j=2}^{\infty} (D_j \cup E_j) \cup Q_1$$

is an open set. Hence  $\text{St}(w, \mathcal{V}_x)$  is open.

Similarly, if  $w \in Q_2$  and  $x \in P^{-1}(Q_2)$ , then  $\text{St}(w, \mathcal{V}_x)$  is again open.

(ii)  $w \in Q_1$  and  $x \notin P(Q_1)$ . Since  $B = Q_1 \cup Q_2 \in \mathcal{V}_x$  (i.e.,  $B \in \mathcal{U}, x \notin B$ ), we see that  $x \notin Q_1$  and so  $x \notin Q_1 \cup P(Q_1)$  and  $Q_1 \cup P(Q_1) \in \mathcal{V}_x$ . The facts that  $Q_1 \cup P(Q_1)$  is open and  $w \in Q_1 \cup P(Q_1) \subseteq \text{St}(w, \mathcal{V}_x)$  are clear. The same conclusion remains valid, if  $w \in Q_2$  and  $x \notin P^{-1}(Q_2)$ . This completes the proof.  $\square$

## 2. Two new cardinal inequalities.

Let  $X$  be a  $T_1$  space. Then we have the following known cardinal inequalities:

$$|X| \leq 2^{e(X)\text{psw}(X)}, \quad (\text{D.K. Burke and R. Hodel [1]}),$$

$$|X| \leq 2^{e(X)\Delta(X)}, \quad (\text{J. Ginsburg and G. Wood [3]}),$$

where

$$\text{psw}(X) = \min\{\kappa \mid \text{there is an open cover } \mathcal{U} \text{ of } X \text{ such that}$$

$$\bigcap \mathcal{U}_x = \{x\}, \quad |\mathcal{U}_x| \leq \kappa, \quad \text{for each } x \in X\},$$

$$\Delta(X) = \min\{\kappa \mid \text{there is a collection of open covers } \{\mathcal{U}_\alpha\}_{\alpha < \kappa}$$

$$\text{of } X \text{ such that } \bigcap \text{St}(x, \mathcal{U}_\alpha) = \{x\} \text{ for each } x \in X\},$$

$$e(X) = \sup\{\kappa \mid A \text{ is a closed discrete subspace of } X \text{ with } |A| \leq \kappa\}.$$

Here  $\kappa$  denotes cardinality and  $|\mathcal{A}|$  denotes the cardinality of  $\mathcal{A}$ .

We will sharpen these inequalities. For this purpose, we define the following cardinal functions:

$$\text{wpsw}(X) = \min\{\kappa \mid \text{there is a cover } \mathcal{U} \text{ of } X \text{ such that } \bigcap \mathcal{U}_x = \{x\},$$

$$|\mathcal{U}_x| \leq \kappa \text{ and } \mathcal{V}_x = \mathcal{U} \setminus \mathcal{U}_x \text{ is }^*\text{-open, for each } x \in X\},$$

$$\overline{\Delta}(X) = \min\{\kappa \mid \text{there is a collection of }^*\text{-open covers } \{\mathcal{U}_\alpha\}_{\alpha < \kappa}$$

$$\text{of } X \text{ such that } \bigcap_{\alpha < \kappa} \text{St}(x, \mathcal{U}_\alpha) = \{x\}, \text{ for each } x \in X\}.$$

Then we have:

**Theorem 1.** For any  $T_1$  space,  $|X| \leq 2^{e(X) \text{wpsw}(X)\psi(X)}$ .

**Theorem 2.** For any  $T_1$  space,  $|X| \leq 2^{e(X)\overline{\Delta}(X)}$ .

To prove our theorems, we need the following results, the first one is easy to prove and the second is due to D.K. Burke.

**Lemma 1.** If  $\mathcal{U}$  is a  $^*$ -open cover of a  $T_1$  space, then there exists a maximal subset  $D$  such that  $x, y \in D$  and  $x \neq y$  imply  $x \notin \text{St}(y, \mathcal{U})$ ; and that  $D$  is a discrete closed subspace of  $X$  with

$$\bigcup_{d \in D} \text{St}(d, \mathcal{U}) = X.$$

**Lemma 2** (D.K. Burke). If  $\{A_\alpha \mid \alpha \in \Lambda\}$  is an indexed collection of sets in which every member has cardinality less than or equal to  $\lambda$ , where  $|\Lambda| > 2^\lambda$ , and each  $A_\alpha$  is a disjoint union of two subsets  $A'_\alpha, A''_\alpha$ , then there is a set  $\Lambda' \subseteq \Lambda$  such that  $|\Lambda'| > 2^\lambda$  and  $A'_\alpha \cap A'_\beta = \emptyset$  whenever  $\alpha, \beta \in \Lambda'$ .

PROOF OF THEOREM 1: Let  $e(X) \text{wpsw}(X)\psi(X) = \kappa$ . Then there is a  $^*$ -open cover  $\mathcal{U}$  of  $X$  such that  $\bigcap \mathcal{U}_x = \{x\}$  and  $|\mathcal{U}_x| \leq \kappa$  for each  $x \in X$ , and a collection of open sets  $\{U_\alpha(x)\}_{\alpha < \kappa}$  such that  $\{x\} = \bigcap_{\alpha < \kappa} U_\alpha(x)$ .

For each  $x_0 \in X$ , we will construct a set

$$A_{x_0} = A'_{x_0} \cup A''_{x_0}$$

satisfying the assumption of Lemma 2. Firstly, since  $|\mathcal{U}_{x_0}| = |\{B \in \mathcal{U} \mid x_0 \in B\}| \leq \kappa$ , we let

$$A'_{x_0} = \mathcal{U}_{x_0}.$$

Then  $\mathcal{V}_{x_0} = \mathcal{U} \setminus \mathcal{U}_{x_0}$  and  $\bigcup \mathcal{V}_{x_0} = X \setminus \{x_0\}$ . For each  $\alpha < \kappa$ , let

$$\mathcal{U}_\alpha = \mathcal{V}_{x_0} \cup \{U_\alpha(x_0)\}.$$

Then  $\mathcal{U}_\alpha$  is a cover of  $X$  such that  $\text{St}(x, \mathcal{U}_\alpha)$  is an open set for each  $x \in X$ . By Lemma 1, there exists a closed subset  $D_\alpha(x_0)$  such that

$$\bigcup \{ \text{St}(d, \mathcal{U}_\alpha) \mid d \in D_\alpha(x_0) \} = X;$$

i.e.,

$$\bigcup \{ \text{St}(d, \mathcal{V}_{x_0}) \mid d \in D_\alpha(x_0) \} \cup U_\alpha(x_0) = X.$$

Since  $e(X) \leq \kappa$ , it follows that  $|D_\alpha(x_0)| \leq e(X) \leq \kappa$ . Therefore

$$\bigcup_{\alpha < \kappa} \bigcup_{d \in D_\alpha(x_0)} \text{St}(d, \mathcal{V}_{x_0}) \supset \bigcup_{\alpha < \kappa} (X \setminus U_\alpha(x_0)) = X \setminus \{x_0\}.$$

On the other hand,

$$x_0 \notin \bigcup_{\alpha < \kappa} \bigcup_{d \in D_\alpha(x_0)} \text{St}(d, \mathcal{V}_{x_0}).$$

Let  $D_{x_0} = \bigcup_{\alpha < \kappa} D_\alpha(x_0)$ . Then we see that

$$\bigcup_{d \in D_\alpha(x_0)} \text{St}(d, \mathcal{V}_{x_0}) = X \setminus \{x_0\} \quad \text{and} \quad |D_{x_0}| \leq \kappa \cdot \kappa = \kappa.$$

Now let

$$A''_{x_0} = \bigcup_{d \in D_\alpha(x_0)} \{B \in \mathcal{V}_{x_0} \mid d \in B\}.$$

Then  $|A''_{x_0}| \leq \kappa \cdot \kappa = \kappa$ . Clearly  $A''_{x_0} \cap A''_{x_0} = \emptyset$ .

If  $|X| > 2^\kappa$ , then by Lemma 2, there is a set  $X' \subseteq X$  such that  $|X'| > 2^\kappa$  and  $A'_x \cap A''_y = \emptyset$  for each pair  $x, y \in X'$ . But this is impossible. Since

$$y \in X \setminus \{x\} = \bigcup_{d \in D_x} \text{St}(d, \mathcal{V}_x),$$

there is a  $B \in \mathcal{V}_x$  and  $d' \in D_x$  such that  $y, d' \in B$  and so  $B \in A'_y \cap A''_x$ ; i.e.,  $A'_y \cap A''_x \neq \emptyset$ , for each distinct pair  $x, y \in X'$ .

Hence  $|X| \leq 2^\kappa$  and the proof is complete. □

**Remark.** We use the technique of Burke in the proof of Theorem 1.

**PROOF OF THEOREM 2:** Let  $e(X)\overline{\Delta}(X) = \kappa$ . Let  $\{\mathcal{W}_\alpha\}_{\alpha < \kappa}$  be a collection of \*-open covers of  $X$  such that  $\bigcap_{\alpha < \kappa} \text{St}(x, \mathcal{W}_\alpha) = \{x\}$ . We will construct an increasing sequence  $\{B_\alpha \mid 0 \leq \alpha < \kappa^+\}$  of subsets in  $X$  and a sequence  $\{\mathcal{U}_\alpha \mid 0 < \alpha < \kappa^+\}$  of open collections in  $X$  such that

(i)  $|B_\alpha| \leq 2^\kappa, 0 \leq \alpha < \kappa^+;$



- (ii)  $\mathcal{U}_\alpha = \bigcup_x \{\text{St}(x, \mathcal{W}_{\alpha'}) \mid \alpha' < \kappa\}$ , where  $x$  runs over the set  $\bigcup_{\beta < \alpha} B_\beta$ , for  $0 < \alpha < \kappa^+$ ;
- (iii) if  $X \setminus (\bigcup \mathcal{U}) \neq \emptyset$ , then  $B_\alpha \setminus (\bigcup \mathcal{U}) \neq \emptyset$ , for each  $\mathcal{U} \in [\mathcal{U}_\alpha]^{\leq \kappa}$ , where

$$[\mathcal{U}_\alpha]^{\leq \kappa} = \{\mathcal{V} \subseteq \mathcal{U}_\alpha \mid |\mathcal{V}| \leq \kappa\}.$$

The construction goes by transfinite induction. Let  $0 < \alpha < \kappa^+$  and assume that  $\{B_\beta \mid \beta < \alpha\}$  have already been constructed. Note that  $\mathcal{U}_\alpha$  is defined by (ii) and  $|\mathcal{U}_\alpha| \leq 2^\kappa$ . For each  $\mathcal{U} \in [\mathcal{U}_\alpha]^{\leq \kappa}$  with  $X \setminus (\bigcup \mathcal{U}) \neq \emptyset$ , choose one point in  $X \setminus (\bigcup \mathcal{U})$ . Let  $A_\alpha$  be the set of all the points chosen in this way. Since  $|\mathcal{U}_\alpha| \leq 2^\kappa$ , it follows that  $|A_\alpha| \leq (2^\kappa)^\kappa = 2^\kappa$ . Now let

$$B_\alpha = A_\alpha \cup \bigcup_{\beta < \alpha} B_\beta.$$

Clearly,  $B_\beta \subseteq B_\alpha$  for all  $\beta < \alpha$ , and  $|B_\alpha| \leq 2^\kappa$ . This completes the construction of the increasing sequence  $\{B_\alpha \mid 0 \leq \alpha < \kappa^+\}$ .

Next, let

$$B = \bigcup_{\alpha < \kappa^+} B_\alpha.$$

Then  $|B| \leq 2^\kappa$ . The proof is complete, if  $X = B$ . Suppose  $X \neq B$  and choose  $p \in X \setminus B$ . For each  $\alpha < \kappa$ , let  $F_\alpha = X \setminus \text{St}(p, \mathcal{W}_\alpha)$ . Then  $F_\alpha$  is closed and

$$\bigcup_{\alpha < \kappa} F_\alpha = X \setminus \bigcap_{\alpha < \kappa} \text{St}(p, \mathcal{W}_\alpha) = X \setminus \{p\} \supseteq B.$$

Let  $\mathcal{V}_\alpha = \{W \in \mathcal{W}_\alpha \mid W \cap (F_\alpha \cap B) \neq \emptyset\}$ . Then we claim that  $\bigcup \mathcal{V}_\alpha \supseteq \overline{F_\alpha \cap B}$ . In fact, if  $y \in \overline{F_\alpha \cap B}$ , then there exists  $b \in \text{St}(y, \mathcal{W}_\alpha) \cap (F_\alpha \cap B)$ , and so  $y \in \text{St}(b, \mathcal{W}_\alpha) \subseteq \bigcup \mathcal{V}_\alpha$ .

Since  $e(X) \leq \kappa$ , we have  $e(\overline{F_\alpha \cap B}) \leq \kappa$ , so that there is a set  $C_\alpha \subseteq F_\alpha \cap B$  such that  $C_\alpha$  is closed discrete with  $|C_\alpha| \leq \kappa$  and

$$\bigcup_{b \in C_\alpha} \text{St}(b, \mathcal{W}_\alpha) = \bigcup_{b \in C_\alpha} \text{St}(b, \mathcal{V}_\alpha) \subseteq F_\alpha \cap B.$$

It is sufficient to take the maximal  $C_\alpha \subseteq F_\alpha \cap B$  such that  $d_1 \notin \text{St}(d_2, \mathcal{V}_\alpha)$  for each distinct pair  $d_1, d_2 \in X$ . Let  $C = \bigcup_{\alpha < \kappa} C_\alpha \subseteq B$ . Then  $|C| \leq \kappa$  and

$$\bigcup_{\alpha < \kappa} \bigcup_{d \in C_\alpha} \text{St}(d, \mathcal{W}_\alpha) \subseteq \bigcup_{\alpha < \kappa} (F_\alpha \cap B) = B.$$

Therefore there exists  $\alpha_0 < \kappa^+$  such that  $C \subseteq B_{\alpha_0}$ . Finally, let

$$\mathcal{U} = \bigcup_{\alpha < \kappa} \{\text{St}(d, \mathcal{W}_\alpha) \mid d \in C_\alpha\}.$$

Then  $\mathcal{U} \in [\mathcal{U}_{\alpha_0}]^{\leq \kappa}$  and hence

$$B_{\alpha_0+1} \setminus (\bigcup \mathcal{U}) \neq \emptyset,$$

by (iii), which is a contradiction. This completes the proof.  $\square$

**Remark.** The results in Section 1 on the concentric circle space show that the above extensions are not trivial.

#### REFERENCES

- [1] Burke D.K., Hodel R., *The number of compact subsets of a topological space*, Proc. Amer. Math. Soc. **58** (1976), 363–368.
- [2] Engelking R., *General Topology*, Warszawa, 1977.
- [3] Ginsburg J., Wood G., *A cardinal inequality for topological space involving closed discrete sets*, Proc. Amer. Math. Soc. **64** (1977), 357–360.

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