Measurable cardinals and category bases

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Abstract. We show that the existence of a non-trivial category base on a set of regular cardinality with each subset being Baire is equiconsistent to the existence of a measurable cardinal.

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The existence of a non-trivial measure $m$ on a set (of regular cardinality) such that each its subset is $m$-measurable, is equiconsistent to the existence of a measurable cardinal [S].

The existence of a non-trivial topology on a set of regular cardinality such that each its subset has the Baire property, is equiconsistent to the existence of a measurable cardinal [KT].

Category bases constitute a common generalization of measure and topological structures. In this case the family of Baire sets plays the role of the $\sigma$-field of measurable sets or the $\sigma$-field of sets with the Baire property. Therefore it is natural to ask about the existence of a non-trivial category base on a set of regular cardinality such that each its subset would be a Baire set. We show that the answer is the same as for the measure or topology, namely, it is equiconsistent to the existence of a measurable cardinal.

Following J.C. Morgan [M1], we say that a pair $(X,C)$ is a category base, if the elements in $C$, called regions, satisfy the following axioms:

1. Every point of $X$ belongs to some region, i.e., $X = \bigcup C$.
2. Let $A$ be a region and let $D$ be any non-empty family of disjoint regions which has the power less than the power of $C$.
   (a) If $A \cap (\bigcup D)$ contains a region, then there is a region $B \in D$ such that $A \cap B$ contains a region.
   (b) If $A \cap (\bigcup D)$ contains no region, then there is a subregion of $A$ which is disjoint from every region in $D$.

As is readily verified, every topology or all subsets of positive measure with respect to a complete $\sigma$-finite measure is a category base. With respect to a given category base, J.C. Morgan has defined abstract versions of the concepts of nowhere dense sets or null sets, sets with the Baire property, and so on.

A set is singular, if every region contains a subregion which is disjoint from it. A countable union of singular sets is called a meager set. A set which is not meager is called an abundant set. A set is abundant everywhere in a region, if every
A subregion of the region intersects the set on an abundant set. A set is a Baire set, if every region contains a subregion, whose intersection with either the set or its complement is meager.

For a given category base \((X, C)\) let \(M(C)\) and \(B(C)\) denote the family of all meager subsets of \(X\) and Baire subsets of \(X\), respectively. In the case that \(C\) is a topology on \(X\), the singular sets coincide with nowhere dense sets, \(M(C)\) coincides with the family of first category sets and \(B(C)\) coincides with the family of sets with the Baire property. In the case that \(C\) is the family of all subsets of positive measure with respect to a \(\sigma\)-finite complete measure, the singular and meager sets coincide with the null-sets and \(B(C)\) coincides with the \(\sigma\)-field of all measurable sets. We are going to utilize also the following basic facts from the theory of category bases.

**Fact 1.** Let \((X, C)\) be a category base. If \(A\) is a region and \(B\) is a Baire set such that \(A \cap B\) is abundant, then there exists an abundant everywhere in itself subregion \(D\) of \(A\) such that \(D - B\) is meager.

**Fact 2.** The intersection of two regions is either a singular set or contains a region.

For proofs of these two particular facts as well as for a more extensive treatment of category bases, we refer to [M1]. The lemma below is new and it constitutes the basic ingredient in the proof of our main theorem.

Let \((X, C)\) be a category base. We say that \(R\) is a category decomposition of \(X\), if \(R\) is a disjoint family of regions such that each region intersects a member of \(R\) on a non-singular set. Notice few elementary facts about category decompositions.

**Fact 3.** Let \(R\) be a category decomposition of \(X\). Then:

(i) \(X - (\bigcup R)\) is singular;
(ii) if \(E \subset X\) and \(E \cap A\) is singular for every \(A \in R\), then \(E\) is singular;
(iii) if \(F \subset X\) and \(F \cap A\) is meager for every \(A \in R\), then \(F\) is meager.

Let \(S\) be a Baire set. An \(M(C)\)-partition of \(S\) is a maximal collection \(W\) of abundant Baire subsets of \(S\) such that \(E \cap F \in M(C)\) for any distinct \(E, F \in W\).

**Lemma.** Let \(S\) be an abundant Baire set. Then for any category decomposition \(R\) of \(X\) and for any \(M(C)\)-partition \(W\) of \(S\), there exists a category decomposition \(Q\) of \(X\) satisfying the following:

(a) every region in \(Q\) is a subregion of a region in \(R\);
(b) if \(A \in Q\) and \(A \cap S\) is abundant, then \(A\) is abundant everywhere in itself and there exists \(B \in W\) such that \(A - B\) is meager.

**Proof:** Let \(C = \{C_\alpha : \alpha < \kappa\}\), where \(\kappa = |C|\). For every \(\beta < \kappa\) we will define a region \(A_\beta\) so that the following are satisfied:

1. the family \(\{A_\alpha : \alpha \leq \beta\}\) is a disjoint family for every \(\beta < \kappa\);
2. \(C_\beta\) intersects some \(A_\alpha\) with \(\alpha \leq \beta\) on a non-singular set;
3. \(A_\beta\) is a subregion of some region from \(R\);
4. if \(A_\beta \cap S\) is abundant, then \(A_\beta\) is abundant everywhere in itself and there exists \(B \in W\) such that \(A_\beta - B\) is meager.
We proceed to define $A_0$. Since $R$ is a category decomposition of $X$, there exists $D \in R$ such that $C_0 \cap D$ is not singular. By virtue of Fact 2, there exists a region $A \subset C_0 \cap D$. If $A \cap S$ is meager, then we set $A_0 = A$. If $A \cap S$ is abundant then, since $W$ is an $M(C)$-partition of $S$, there exists $B \in W$ such that $A \cap B$ is abundant. According to Fact 1, there exists a subregion $A'$ of $A$ which is abundant everywhere in itself and yet $A' - B$ is meager. We then set $A_0$ to be $A'$.

Let $\beta < \kappa$ and suppose $A_\alpha$ has been well defined for every $\alpha < \beta$. If $C_\beta \cap A_\alpha$ is non-singular for some $\alpha < \beta$, then we set $A_\beta = A_0$. If $C_\beta \cap A_\alpha$ is singular for each $\alpha < \beta$, then there exists a subregion $A$ of $C_\beta$ disjoint with every $A_\alpha, \alpha < \beta$. We may assume that $A$ is also a subregion of some region from $R$. If $A \cap S$ is meager, then we set $A_\beta = A$. If $A \cap S$ is abundant, then, arguing as in the first part, there exist $B \in W$ and a subregion $A'$ of $A$ such that $A'$ is abundant everywhere in itself and $A' - B$ is meager. Then we set $A_\beta = A'$.

It follows immediately from the conditions (1)-(4) that the family $Q = \{A_\alpha : \alpha < \kappa\}$ is a category decomposition of $X$ satisfying the conditions (a) and (b).

Now we will need some facts about ideals. Let $\kappa$ be an infinite cardinal. A set $I$ of subsets of $\kappa$ is an ideal over $\kappa$, if

(i) $\{\alpha\} \in I$ for each $\alpha \in \kappa$ and $\kappa \notin I$.
(ii) If $X \in I$ and $Y \subset X$, then $Y \in I$.
(iii) If $X \in I$ and $Y \in I$, then $X \cup Y \in I$.

A $\sigma$-complete ideal is an ideal which is closed under countable unions. If $(X, C)$ is a non-trivial category base in the sense that one-point sets are singular and the set $X$ is abundant, then $M(C)$ is a $\sigma$-ideal over $|X|$. Customarily it is convenient to adopt the following terminology about ideals. Let $I$ be an ideal over $\kappa$. Elements of $I$ are called sets of $I$-measure zero; elements of $I^+ = P(\kappa) - I$ are called sets of positive $I$-measure; elements of $I^* = \{\kappa - x : x \in I\}$ are called sets of $I$-measure one; when the context is clear, we drop the prefix “$I$-” in all above phrases.

Let $S$ be a set of positive measure. An $I$-partition of $S$ is a maximal collection $W$ of subsets of $S$ of positive measure such that $X \cap Y \in I$ for any distinct $X, Y \in W$. An $I$-partition $W_1$ of $S$ is a refinement of an $I$-partition $W_2$ of $S$, $W_1 \leq W_2$, if every element of $W_1$ is a subset of an element of $W_2$.

The ideal $I$ is weakly precipitous, if $I$ is $\sigma$-complete and whenever $S$ is a set of positive measure and $\{W_n : n \in \omega\}$ are $I$-partitions of $S$ such that $W_0 \geq W_1 \geq \cdots \geq W_n \geq \cdots$, then there exists a sequence $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$ such that $X_n \in W_n$ for each $n$, and $\bigcap\{X_n : n \in \omega\} \neq \emptyset$. The following is of a basic importance for us.

**Fact 4.** If a regular uncountable cardinal $\kappa$ carries a weakly precipitous ideal, then $\kappa$ is measurable in some transitive model of ZFC.

This strong theorem is due to T. Jech, M. Magidor, W. Mitchell and K. Prikry [JP]. Originally, the theorem was formulated for precipitous ideals but the proof works for weakly precipitous ideals as well. For more extensive treatment about ideals, we refer to Jech’s book [J].
Theorem. CON (ZFC + “there exists a non-trivial category base \((X, C)\) such that \(|X|\) is a regular cardinal and \(B(C) = P(X)\)”), if and only if CON (ZFC + “there exists a measurable cardinal”).

Proof: Assume first the consistency of the existence of a measurable cardinal. According to a theorem of Solovay [S], it implies the consistency of the existence of a set \(X\) of regular cardinality not exceeding the power of the continuum and a probabilistic non-trivial measure \(m\) on \(P(X)\). Hence the family \(C\) of all sets of positive \(m\)-measure is a non-trivial category base on \(X\) such that \(B(C) = P(X)\).

To prove the converse implication, we rely on Fact 4 and we will try to discover a weakly precipitous ideal. In fact, if \((X, C)\) is a non-trivial category base such that \(|X|\) is a regular cardinal and \(B(C) = P(X)\), then \(M(C)\) is weakly precipitous.

Clearly, \(M(C)\) is \(\sigma\)-complete. Let \(S \subseteq X\) be a set of positive measure and let \(\{W_n: n \in \omega\}\) be an appropriate sequence of partitions of \(S\): We define inductively a sequence \(\{Q_n: n \in \omega\}\) of category decompositions of \(X\). We set \(Q_0\) as a category decomposition of \(X\) obtained by an application of Lemma for \(R = \{X\}\) and \(W = W_0\). We set \(Q_{n+1}\) as a category decomposition of \(X\) obtained by an application of Lemma for \(R = Q_n\) and \(W = W_{n+1}\). Let \(Q_n^* = \{A \in Q_n : A \cap S\) is abundant \} and let \(B(A)\) denote a unique element in \(W_n\) such that \(A - B(A)\) is meager whenever \(A \in Q_n^*\).

Observe that both sets \(E_n = S - (\bigcup Q_n^*)\) and \(F_n = \bigcup \{A - B(A) : A \in Q_n^*\}\) are meager for each \(n \in \omega\). To see this, notice that both \(E_n\) and \(F_n\) intersect every member of the family \(Q_n\) on a meager set. Since \(Q_n\) is a category decomposition of \(X\), they are meager, according to Fact 3 (iii).

Consider now the set \(S^* = S - (\bigcup \{E_n \cup F_n : n \in \omega\})\). Since \(S\) is abundant, \(S^*\) is not empty and let \(p\) be its arbitrary point. Since \(S^* \subseteq S - (\bigcup \{E_n : n \in \omega\}) = S \cap \bigcap \{Q_n^* : n \in \omega\}\), for each \(n \in \omega\) there exists a unique element \(A_n\) of \(Q_n^*\) containing the point \(p\). Let \(B_n = B(A_n)\) be a unique element of the partition \(W_n\) such that \(A_n - B_n\) is meager. Since \(A_n - B_n \subseteq F_n, p \in B_n\) for every \(n \in \omega\). At the end let us prove that the selected sets \(B_n\) form a decreasing sequence. Notice at first that this fact is true about the sequence \(A_n\), since each \(Q_n\) is a disjoint family being a refinement of the preceding one. Let \(B \in W_{n-1}\) be such that \(B_n \subseteq B\). If \(B\) were different from \(B_{n-1}\), then \(B \cap B_{n-1}\) would be meager. Since \(A_n - B_{n-1}\) is meager, \(B \cap A_n\) is meager and so is \(B \cap A_n\). From the other side, every \(A_n\) is abundant and \(A_n - B_n\) is meager. Since \(B_n \subseteq B, B \cap A_n\) would be abundant; a contradiction. \(\square \)

References


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