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A New Semi-Orthogonal Relation for the Laguerre Polynomials

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Abstract. A new semi-orthogonal relation for the Laguerre polynomials is given with an elementary weight function.

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1. Introduction

The Laguerre polynomials are orthogonal polynomials [1, p.183, (16) and (17)] over the interval (0, \(\infty\)) with respect to the weight function \(e^{-x}x^a\), if \(\text{Re}a > -1\).

In this paper, we present a new semi-orthogonal relation for the Laguerre polynomials over the interval (0, \(\infty\)) with respect to the weight function \(e^{-x}x^{n-m+a-1}\), if \(\text{Re}a > m-n\). With the help of our semi-orthogonal relation, we obtain a Fourier-Laguerre expansion for an elementary function.

The Laguerre polynomials are defined by the relation [1, p.325, 6(a)]:

\[
L_n^a(x) = \frac{(-1)^n}{n!}x^n e^{-x} \frac{\Gamma(a+n)}{\Gamma(a)} \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} \tag{1.1}
\]

2. The Semi-Orthogonal Relation

The semi-orthogonal relation to be established is

\[
\int_0^\infty e^{-x}x^{n-m+a-1} L_m^a(x) L_n^a(x) \, dx = 0, \quad \text{if } m < n \tag{2.1a}
\]

\[
= \frac{\Gamma(a)(a+1)}{n!}, \quad \text{if } m = n \tag{2.1b}
\]

\[
= \frac{2\Gamma(a-1)(a+2)}{n!}, \quad \text{if } m = n + 1 \tag{2.1c}
\]

where \(\text{Re}a > m - n\).
PROOF: In view of (1.1), the integral (2.1) can be written as

\[
\frac{(-1)^{m+n}}{m!n!} \int_0^\infty e^{-x} x^{2n+a-1} \, \sum_{r=0}^m \frac{(-m)_r (-m-a)_r (-1)^r}{r!} x^n \, dx =
\]

\[
= \frac{(-1)^{m+n}}{m!n!} \sum_{r=0}^m \frac{(-m)_r (-m-a)_r (-1)^r}{r!} \times
\]

\[
\times \sum_{u=0}^n \frac{(-n)_u (-n-a)_u (-1)^u}{u!} \times
\]

\[
\times \int_0^\infty e^{-x} x^{2n+a-1-r-u} \, dx
\]

(2.2)

Evaluating the last integral in (2.2) with the help of the definition of the gamma-function [1, p.335, (1)], then using the relation [1, p.275, (8)], viz.

\[
\Gamma(a + 1 - n) = \frac{(-1)^n \Gamma(a+1)}{(-a)_n} \text{ and simplifying,}
\]

the right hand side of (2.2) becomes

\[
\frac{(-1)^{m+n}}{m!n!} \sum_{r=0}^m \frac{(-m)_r (-a-m)_r (-1)^r \Gamma(2n+a-r)}{r!(1-2n-a+r)_n} \, \sum_{u=0}^n \frac{(-n)_u (-n-a)_u (-1)^u}{u!} (2.3)
\]

Now applying Vandermode's theorem [1, p.283, 19(a)], viz.

\[
F(-n, a; c, 1) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, 2, \ldots
\]

(2.4)

to (2.3) and using the relation \((1-n+r)_n = (-1)^n (-r)_n\), we have

\[
\frac{(-1)^{m+n}}{m!n!} \sum_{r=0}^m \frac{(-m)_r (-r)_n (-a-m)_r \Gamma(2n+a-r) (-1)^{r+n}}{r!(1-2n-a+r)_n} \, \sum_{u=0}^n \frac{(-n)_u (-n-a)_u (-1)^u}{u!} (2.5)
\]

If \(r < n\), the numerator of (2.5) vanishes, and since \(r\) runs from 0 to \(m\), it follows that (2.5) also vanishes, when \(m < n\). Now, it is clear that for \(m < n\) all terms of (2.5) vanish, which proves (2.1a).

When \(m = n\), using the standard result

\[
(-r)_n = \begin{cases} 
\frac{(-1)^r r!}{(r-n)!}, & \text{if } 0 \leq n \leq r \\
0, & \text{if } n > r
\end{cases}
\]

(2.6)

and simplifying, we have

\[
\int_0^\infty e^{-x} x^{a-1} \, (L_n^a(x))^2 \, dx = \frac{\Gamma(a)(a+1)_n}{n!} \quad \text{Re}a > 0,
\]

(2.7)
which proves (2.1b).

In (2.5), putting \( m = n + 1 \), using (2.6) and adding the resulting two terms \((r = n, n + 1)\), and simplifying, we obtain

\[
\int_0^\infty e^{-x} x^{a-2} L_{n+1}^a(x) L_n^a(x) \, dx = \frac{2\Gamma(a-1)(a+2)_n}{n!}, \quad \text{Re}a > 1
\]

which proves (2.1c).

\[\square\]

**Note.** On continuing as above we can find the values of the integral (2.1) for \( m = n + 2, n + 3, n + 4, \ldots \)

### 3. Fourier-Laguerre Expansion

Based on the relations (2.1a) and (2.1b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in a finite series expansion of the Laguerre polynomials. Specially if \( f(x) \) is a suitable function defined for all \( x \), we consider for expansions of the general form

\[
f(x) = \sum_{m=0}^{n} C_m x^{-m} L_m^a(x), \quad 0 < x < \infty, \quad m < n
\]

where the Fourier coefficients \( C_m \) are given by

\[
C_m = \frac{m!}{\Gamma(a)(a+1)_m} \int_0^\infty e^{-x} x^{m+a-1} f(x) L_m^a(x) \, dx
\]

### 4. Fourier-Laguerre Expansion For \( x^{-n} \)

The Fourier-Laguerre expansion to be obtained is

\[
f(x) = x^{-n} = \frac{1}{\Gamma(a)} \sum_{m=0}^{n} (-1)^m (-n)_m \frac{\Gamma(m-n+a)}{(a+1)_m} x^{-m} L_m^a(x)
\]

where \( \text{Re}a > n - m \).

**Proof:** On using the following modified form of the integral [2, p.292, (1)]:

\[
\int_0^\infty x^{b-1} e^{-x} L_m^a(x) \, dx = (-1)^n \frac{\Gamma(b) \Gamma(b-a)}{n! \Gamma(b-a-m)}
\]

where \( \text{Re}b > 0 \).

and (3.1) and (3.2) with \( f(x) \) given in (4.1), the Fourier-Laguerre expansion (4.1) is obtained.

\[\square\]
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