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THE EXISTENCE OF A CONTINUOUS BASIS OF A CERTAIN LINEAR SUBSPACE OF $E_r$ WHICH DEPENDS ON A PARAMETER

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In the article a theorem concerning the existence of a continuous basis of the space of all solutions $x \in E_r$ of the equation $A(t)x = 0$ is given.

Let $A(t)$ be an $r \times r$ matrix which is continuous on $(0, \infty)$ and let $S_t \subset E_r$ be the linear space of all solutions $x$ of the equation $A(t)x = 0$ for a chosen $t \geq 0$; the question is whether there is a fixed set of continuous vectors $P_t = \{x_1(t), x_2(t), \ldots, x_k(t)\}$ such that $P_t$ is a basis of $S_t$ for any $t \geq 0$. The answer is contained in the following theorem:

**Theorem.** Let $A(t)$ be an $r \times r$ matrix which has a continuous $n$-th derivative everywhere in $(0, \infty)$, $n \geq 0$; moreover, let an integer $h < r$ exist such that $\text{rank} \ A(t) = h$ for every $t \in (0, \infty)$. Then there is an $r \times r$ matrix $M(t)$ which possesses a continuous $n$-th derivative in $(0, \infty)$ such that $\det M(t) \neq 0$ in $(0, \infty)$ and $A(t)M(t) = [B(t) \mid 0]$, where $B(t)$ is an $r \times h$ matrix with $\text{rank} B(t) = h$ for every $t \in (0, \infty)$.

Obviously, the last $r - h$ columns of the matrix $M(t)$ constitute the sought set $P_t$.

**Proof.** Choose a $\bar{T} > 0$. Since $A(t)$ is continuous, a minor of $A(t)$ with order $h$ exists which is different from zero on an interval $(0, \delta)$. By the same argument, for each $t \in (\delta/2, \bar{T})$ there is an open interval $J_t$ containing $t$ such that a minor of $A(t)$ with order $h$ exists which is different from zero on $J_t$. The system of all intervals $\{J_t\}$, $t \in (\delta/2, \bar{T})$, however, covers $(\delta/2, \bar{T})$; consequently, by Borel’s theorem, there is a finite subsystem $\{J_1, J_2, \ldots, J_k\}$ of $\{J_t\}$ with the same property. From this it follows that there is a sequence of closed intervals $I_i = (t_i, t_i^*)$, $i = 1, 2, \ldots$ which has the properties:

a) $t_1 = 0$, $t_i < t_{i+1} < t_i^* < t_{i+1}^*$, $i = 1, 2, \ldots$, $t_i \to \infty$,

b) for every $i$ there is a minor $A_i(t)$ of the matrix $A(t)$ with order $h$ such that $|\det A_i(t)| \geq c_i > 0$ for $t \in I_i$. 


Using this fact it can be easily verified that for every $i = 1, 2, \ldots$ there is an $r \times r$ matrix $M(t)$ such that

1) $M(t)$ is defined on $I_i$, possesses a continuous $n$-th derivative there and $\det M(t) = \tilde{c}_i \neq 0$ on $I_i$,

2) $A(t) M(t) = [B(t) \mid 0]$, where $B(t)$ is an $r \times h$ matrix with rank $B(t) = h$ on $I_i$.

Indeed, for every $i$ there are constant regular $r \times r$ matrices $C, D$ such that

\[ C(t) \frac{D(t)}{A(t)} = \left[ \begin{array}{c} A_{11}(t) \ A_{12}(t) \\ A_{21}(t) \ A_{22}(t) \end{array} \right] \]

where $A_{11}(t)$ is an $h \times h$ matrix fulfilling the inequality $|\det A_{11}(t)| \geq c_i > 0$ for every $t \in I_i$. Thus putting

\[ M(t) = D(t) \left[ \begin{array}{c} I \ (-A_{11}(t))^{-1} A_{12}(t) \\ 0 \ I \end{array} \right] \]

where $I$ denotes the unit matrix, we can verify that the matrix $M(t)$ has the properties stated above.

Consider now two neighboring intervals $I_i$ and $I_{i+1}$. Denoting $K_i = (t_{i+1}, t_i) \subset I_i \cap I_{i+1}$, choose a number $\tau_i \in K_i$. Then we have $A(\tau_i) M(\tau_i) = [B(\tau_i) \mid 0]$, $A(\tau_i) M_{i+1}(\tau_i) = [B_{i+1}(\tau_i) \mid 0]$, consequently, there is a constant regular $r \times r$ matrix $F_i$ such that

\[ M(\tau_i) = M_{i+1}(\tau_i) F_i, \]

and $F_i$ has the form

\[ F_i = \left[ \begin{array}{c} F_{11}^{(i)} \ 0 \\ F_{21}^{(i)} \ F_{22}^{(i)} \end{array} \right], \]

$F_{11}^{(i)}$ being an $h \times h$ matrix.

Let $\eta(t)$ be a function which possesses a continuous $n$-th derivative on $K_i$ and fulfills the inequality $0 \leq \eta(t) \leq 1$, $t \in K_i$, and define the matrix $H(t)$ on $K_i$ by

\[ H(t) = M(t) + \eta(t) (M_{i+1}(t) F_i - M(t)). \]

Obviously, $H(t)$ has a continuous $n$-th derivative on $K_i$ and due to the form of $F_i$ we have $A(t) H(t) = [\tilde{B}(t) \mid 0]$ on $K_i$, $\tilde{B}(t)$ being an $r \times h$ matrix. Moreover, $H(\tau_i) = M(\tau_i)$.

Next, denoting the elements of $M(\tau_i)$ by $m_{jk}(\tau_i)$, $j, k = 1, 2, \ldots, r$, consider the expression

\[ \Phi(t, \xi) = |\det [m_{jk}(t) + \xi_{jk}]| \]

as a function of $r^2 + 1$ variables $t \in K_i$ and $\xi_{jk} \in (-a, a)$, $j, k = 1, 2, \ldots, r$. Then we have $\Phi(\tau_i, 0) = |\det M(\tau_i)| = |\tilde{c}_i| \neq 0$. Since $\Phi(t, \xi)$ is a continuous function of
its variables, there is an open interval $K_t \subset K_1$ which contains $\tau_i$ and a number $\delta > 0$ such that

$$|\xi| < \varphi(t, \xi) < \frac{3|\xi|}{2}$$

for every $t \in K_t$ and $\xi \in (-\delta, \delta)$, $j, k = 1, 2, \ldots, r$.

On the other hand, since the matrix $Q_i(t) = M_{i+1}(t) F_i - M_i(t)$ is continuous on $K_i$, there is an open interval $K_t^* \subset K_t$ containing $\tau_i$ such that for every element $q_{jk}(t)$, $j, k = 1, 2, \ldots, r$ of $Q_i(t)$ we have $|q_{jk}(t)| < \delta$ whenever $t \in K_t^*$. Consequently, using (2), we have

$$|\xi| < |\det H(t)| < \frac{3|\xi|}{2}$$

for every $t \in K_t \cap K_t^*$.

Thus, denote $K_t \cap K_t^* = (t_{i+1}, \tilde{t}_i)$ and choose a function $\eta(t)$ which has a continuous $n$-th derivative and satisfies the conditions $\eta(t) = 0$ for $t \in (t_i, t_{i+1})$, $0 < \eta(t) < 1$ for $t \in (\tilde{t}_i, \tilde{t}_i)$, $\eta(t) = 1$ for $t \in (t_i, t_{i+1})$. Putting then

$$H(t) = (1 - \eta(t)) M(t) + \eta(t) M_{i+1}(t) F_i,$$

where $M_k(t) = M_k(t)$ on $I_k$, $M_k(t) = 0$ elsewhere, $k = i, i + 1$, the matrix $H(t)$ is defined on the entire interval $<t_i, t_{i+1}> = I_i \cup I_{i+1}$, possesses a continuous $n$-th derivative there and by (5) fulfills the conditions $\det H(t) \neq 0$, $A(t) H(t) = = [B(t) \mid 0]$ on $I_i \cup I_{i+1}$, where $B(t)$ is an $r \times h$ matrix.

From the above considerations it follows that there is a sequence of closed intervals $I_i = \langle i, \tilde{i}_i \rangle$, $i = 1, 2, \ldots$, where $I_i < I_l$, $\tilde{i}_i = 0$, $t_i < \tilde{t}_{i+1} < \tilde{t}_i$, $t_i < \tilde{t}_{i+1}$, $i = 1, 2, \ldots$, $\tilde{t}_i \to \infty$, which has the following property: Defining successively matrices $M_i(t)$ on $<0, \infty>$ by

$$M_1(t) = M_1(t)$$

$$M_{i+1}(t) = M_{i+1}(t) F_i$$

$$= 0$$

$$\text{elsewhere}, \quad = 0$$

elsewhere,

$i = 1, 2, \ldots$, where each matrix $F_i$ can be obtained from matrices $M_i(t)$, $M_{i+1}(t)$, $\tau_i \in I_i \cap I_{i+1}$ as indicated above, and functions $\tilde{\eta}_i(t)$, $i = 1, 2, \ldots$ with a continuous $n$-th derivative by $\tilde{\eta}_i(t) = 1$ on $<0, \tilde{t}_i>$, $0 < \tilde{\eta}_i(t) < 1$ on $(\tilde{t}_i, \tilde{t}_{i+1})$, $\tilde{\eta}_i(t) = 0$ on $<\tilde{t}_i, \infty>$, and

$$\tilde{\eta}_i(t) = 1$$

on $<\tilde{t}_{i-1}, \tilde{t}_{i+1}>$, $0 < \tilde{\eta}_i(t) < 1$ on $(\tilde{t}_{i+1}, \tilde{t}_{i}),$

$$\tilde{\eta}_i(t) + \tilde{\eta}_{i-1}(t) = 1$$

on $(\tilde{t}_i, \tilde{t}_{i-1})$ and $\tilde{\eta}_i(t) = 0$ elsewhere,

then the matrix

$$M(t) = \sum_{i=1}^{\infty} \tilde{\eta}_i(t) M_i(t)$$

has all the properties stated in the Theorem.

The assertion that $\text{rank } B(t) = h$ is obvious; hence, the Theorem is proved.
Résumé

EXISTENCE SPOJITÉ BÁZE JISTÉHO LINEÁRNÍHO PODPROSTORU $E_r$, ZÁVISLÉHO NA PARAMETRU

VÁCLAV DOLEŽAL, Praha

V článku je dokázána věta o tom, že ke každé čtvercové matici $A(t)$, která je spojitá a má pevnou hodnotu na intervalu $<0, \infty)$, existuje pevná soustava spojitých vektorů $P_t = \{x_1(t), x_2(t), \ldots, x_k(t)\}$ tak, že pro každé $t \geq 0$ je $P_t$ bazí podprostoru všech řešení rovnice $A(t)x = 0$.

Резюме

СУЩЕСТВОВАНИЕ НЕПРЕРЫВНОГО БАЗИСА НЕКОТОРОГО ЛИНЕЙНОГО ПОДПРОСТРАНСТВА $E_r$, ЗАВИСЯЩЕГО ОТ ПАРАМЕТРА

ВАЦЛАВ ДОЛЕЖАЛ (Václav Doležal), Прага

В статье доказывается теорема о том, что для каждой квадратной матрицы $A(t)$, которая непрерывна и имеет фиксированный ранг на интервале $<0, \infty)$, существует фиксированная система непрерывных векторов $P_t = \{x_1(t), x_2(t), \ldots, x_k(t)\}$ так, что для любого $t \geq 0$ система $P_t$ является базисом подпространства всех решений уравнения $A(t)x = 0$. 