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COURT'S CONJECTURE ON $n + 2$ POINTS IN $[n]$

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1. Conjecture. Let A_i $(i = 0, 1, \ldots, n + 1)$ be $n + 2$ distinct points in an $[n]$ such that no $n + 1$ of them lie in a prime or hyperplane and therefore any $n + 1$ then form the $n + 1$ vertices A_j of a simplex $S(A_i)$ $(j \neq i)$ and p_i be the harmonic polar, or simply polar, prime of A_i w.r.t. $S(A_i)$ as defined and used in several earlier works ([7]; [8]; [11]; [13]; [14]; [16]–[22]; [24]; [25]; [27]). The Court's conjecture is [3]: "*The $n + 2$ primes p_i are such that any $n + 1$ of them form the $n + 1$ primes p_j of a simplex $S(p_i)$ $(j \neq i)$ perspective to $S(A_i)$ formed by their $n + 1$ corresponding points A_j. The centre and prime of perspectivity are in each case the remaining point A_i and the remaining prime p_i. The constants of the $n + 2$ perspectivities considered are equal, their common numerical value being $n + 2$.*" (cf. [2a]).

2. Proof. It is just a proposition of incidence alone and can be easily established by using symbols of the points A_i by the same letters as used by Baker ([2], p. 115), Coxeter [4], Room [5] and Mandan ([6]; [7]; [9]; [12]; [23]; [26]). These symbols, since the $n + 2$ points are in $[n]$, must be connected by a syzygy, which, by proper choice of the symbols, may be supposed to be $\sum A_i \equiv 0$. Thereafter no further multiplication of these symbols by an algebraic symbol is legitimate, save by one the same for all. We suppose the $n + 2$ symbols not to be connected by any further syzygy, the $n + 2$ points not being in an $[n - 1]$ (cf. [15]).

The polar prime p_i of a point A_i w.r.t. the simplex $S(A_i)$ then ([6]; [7]) contains the $\binom{n+1}{2}$ points $A_j - A_k$ or $A_k - A_j (j, k \neq i)$, and is determined by any n independent points of the type by fixing either j or k. Thus the $\binom{n+2}{2}$ points $A_i - A_j$ or $A_j - A_i$ lie by $\binom{n+1}{2}$s in the $n + 2$ primes p_k such that just n primes pass through each point, and the $n + 1$ primes $p_j (j \neq i)$ form a simplex $S(p_i)$ whose $n + 1$ opposite vertices are the $n + 1$ points $A_i - A_j$. Hence the simplex $S(A_i)$ with vertices at the $n + 1$ points $A_j (j \neq i)$ is obviously perspective to $S(p_i)$ with vertices at $A_i - A_j$ from the point A_i as their centre of perspectivity. Again the edges

A_jA_k of $S(A_i)$ meet the corresponding edges joining the corresponding vertices $A_i - A_j$, $A_i - A_k (j, k \neq i)$ which lie in the prime p_i by definition thus giving us the desired prime of perspectivity of the two simplexes under consideration.

The join of the points A_i, A_j meets the prime p_i in the point $A_i + mA_j$ such that $A_i + mA_j$ is a linear relation of n independent points $A_j - A_k$ (with j fixed) of p_i. That is, $A_i + mA_j \equiv \sum_k m_k(A_j - A_k)$. This identity must reduce to the given syzygy $\sum A_i \equiv 0$. Hence $m_k = 1$ and $m = n + 1$. The cross or anharmonic ratio or simply biratio of the 4 points A_i, A_j, $A_i + mA_j$, $A_i - A_j$, called the constant of perspectivity of the 2 perspective simplexes $S(A_i)$ and $S(p_i)$, is therefore given by their parameters 0, ∞, m, -1 when considered as the 4 points $A_i + rA_j$ with r as their parameter, and is then found to be equal to $m + 1 = n + 2$ as required.

3. Dual. The dual proposition would now run as follows: "*Given $n + 2$ primes in an $[n]$, if for each one we construct the harmonic pole w.r.t. the simplex determined by the remaining $n + 1$ primes, the $n + 2$ points thus obtained are such that any $n + 1$ of them and their $n + 1$ corresponding primes form 2 simplexes which are in perspective. The centre and prime of perspectivity are the remaining point and the remaining prime, the constant of perspectivity in each case being equal to $n + 2$.*

4. Orthocentric group. The vertices of an orthogonal simplex and its orthocentre in an $[n]$ form an orthocentric group of $n + 2$ points such that any $n + 1$ of them form an orthogonal simplex with the remaining point as its orthocentre ([19]; [21]; [22]; [24]). The orthic axes of the triangles of an orthogonal simplex lie in the polar prime, called its orthic prime, of its orthocentre w.r.t. it [22]. Hence we have

Theorem 1. *The simplex formed of any $n + 1$ of an orthocentric group of $n + 2$ points in an $[n]$ is perspective to that formed by the $n + 1$ orthic primes of the remaining $n + 1$ orthogonal simplexes formed of the given group of points, the centre and prime of perspectivity being its orthocentre and orthic prime. The constant of perspectivity is equal to $n + 2$.*

5. Special case. When the $n + 2$ points of the conjecture are such that one of them lies at the centroid of the simplex formed by the rest, the proposition becomes

Theorem 2. *Given a simplex, if for each of its vertices the polar prime is constructed w.r.t. the simplex formed of its n remaining vertices and its centroid, the $n + 1$ primes thus obtained form a simplex homothetic to it, and the two simplexes have the same centroid. Their homothetic ratio is equal to $n + 2$.*

For the polar prime of the centroid of a simplex w.r.t. it lies at infinity, and when a pair of perspective simplexes have their prime of perspectivity at infinity, they become homothetic and their constant of perspectivity becomes their homothetic ratio ([3]; [13]; [14]).

6. Self-conjugate $(n + 2)$ ads. We know ([1], pp. 36–46; [2], pp. 148–149; [10]) that a pair of perspective simplexes are polar reciprocal for a unique quadric for which their centre and prime of perspectivity are pole and polar. Analytically if we take the simplex (S) formed by any $n + 1$ points of the conjecture as one of reference and the remaining point as its unit point, the equation of the quadric, for which the primes p_i are respectively the polars of the points A_i, can be easily obtained ([1], p. 46) as

$$(n + 2)(\sum x_i^2) = (\sum x_i)^2 \quad (i = 0, 1, \ldots, n).$$

This is a particular case for $a_i = 1$ of the equation

$$(1 + \sum a_i)(\sum a_i x_i^2) = (\sum a_i x_i)^2$$

which represents a quadric such that (S) reciprocates into a simplex whose vertex corresponding to its vertex A_i has its ith coordinate as $x_i = 1 + 1/a_i$, and the remaining n coordinates being all $x_j = 1$ $(j \neq i)$.

The equation of the prime p_i other than that corresponding to the unit point (its equation being that of the unit prime $\sum x_i = 0$) is easily found to be $\sum x_i = (n + 2)\, x_i$. Hence we have [10].

Theorem 3. *There always exists a unique quadric Q for which the given $n + 2$ points A_i in $[n]$ form a selfconjugate $(n + 2)$ad such that the pole for Q of the prime determined by any n of them lies on the join of the remaining two, and the corresponding $n + 2$ primes p_i of the conjecture too form a dual self-conjugate $(n + 2)$ ad for Q such that the polar prime for Q of the point determined by any n of them passes through the $[n - 2]$ common to the remaining two.*

7. Remarks. The method of symbols adopted above can also be now usefully used to re-establish certain results in the case of cevian simplexes [13].

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Výtah

COURTOVA DOMNĚNKA O $n + 2$ BODECH V $[n]$

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V článku je dokázána domněnka, kterou o perspektivnosti dvojic určitých simplexů vyslovil N. A. Court. Výsledek je ještě dualisován a aplikován na případ ortocentrických skupin bodů v euklidovském prostoru.

Резюме

ПРЕДПОЛОЖЕНИЕ КУРТА О $n + 2$ ТОЧКАХ В $[n]$

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В статье доказано предположение, высказанное Н. А. Куртом (N. A. Court) и касающееся перспективности пар определенных симплексов. Кроме того, доказан еще двойственный результат, и показано приложение к случаю ортоцентрических систем точек в евклидовом пространстве.