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GEOMETRIC STRUCTURE WITH HILBERT'S AXIOMS OF INCIDENCE AND ORDER

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1. According to well known results [3, chap. 5, 6], the projective axioms of incidence are naturally induced in a projective three-dimensional number space $\mathbf{P}(\mathbf{K})$ over any skew-field $\mathbf{K}$, and conversely, for any geometric projective space $\mathbf{P}$ there exists a uniquely determined skew-field $\mathbf{K} = \mathbf{K}(\mathbf{P})$ such that $\mathbf{P}$ is isomorphic to the number space $\mathbf{P}(\mathbf{K})$ with respect to the relation of incidence. If $\mathbf{P}$ is an ordered projective space, i.e. the relation of division (or separation) [5, p. 365] is introduced in $\mathbf{P}$, then the skew-field $\mathbf{K}(\mathbf{P})$ may be ordered, and conversely, if $\mathbf{K}$ is an ordered skew-field, then the relation of division is naturally induced in $\mathbf{P}(\mathbf{K})$. Finally, if the Dedekind axiom holds in $\mathbf{P}$, then $\mathbf{K}(\mathbf{P})$ is the field of all real numbers, and also conversely. This bijection between projective spaces and their algebraic models may sometimes be used for solving of some geometric problems.

In this paper, we shall investigate analogous models for structures with Hilbert's axioms of incidence and order. We shall use Pasch's construction of projective extension introduced for another purpose in [1]. The terminology and notation in this paper are the same as in [5], as far as possible. In particular, $\mathbf{L}(ab)$ denotes the line determined by distinct points a and b, and $\mathbf{B}(a, b, c)$ means that the point b lies between the points a and c.

2. A set $\mathbf{S}$ (of points) with two classes of subsets (lines and planes) and a ternary relation $\mathbf{B}$ (the betweenness relation) satisfying Hilbert's axioms of incidence and order (cf. [2] or [5, chap. I]) will be termed an *ordered incidence space.*

A subset $\mathbf{C} \subset \mathbf{S}$ is called a *convex set,* if

(a) $\mathbf{C}$ is open (the definition of open sets is analogous to [5, p. 64]),

(b) if $a, b \in \mathbf{C}$ and $\mathbf{B}(a, c, b)$, then $c \in \mathbf{C}$.

The following theorem is easily verified.

Theorem 1. *Let $\mathbf{C}$ be a convex set in $\mathbf{S}$. If lines and planes in $\mathbf{C}$ are introduced as non-empty intersections of lines and planes in $\mathbf{S}$ with $\mathbf{C}$, and if the betweenness relation on $\mathbf{C}$ is defined by restriction, then $\mathbf{C}$ is an ordered incidence space.*

3. Now, let $\boldsymbol{P}$ be an ordered projective space. On choosing a plane P_∞ in $\boldsymbol{P}$, one may introduce on $\boldsymbol{A} = \boldsymbol{P} \setminus P_\infty$ the structure of an ordered incidence space in the classical manner [5, p. 370]. Then $\boldsymbol{A}$ is called an affine space over the skew-field $\boldsymbol{K}(\boldsymbol{P})$. By theorem 1 we have

Theorem 2. *Every convex set in an affine space over an ordered skew-field is an ordered incidence space.*

4. In this section, we shall show that every ordered incidence space $\boldsymbol{S}$ may be naturally imbedded into an ordered projective space. The correctness of following definitions and the proofs of following assertions may be found in [1, §3–§9].

Let $L_1 \neq L_2$ be lines in $\boldsymbol{S}$ lying in the same plane P. The bundle determined by them is defined as the set of all lines L with the following properties:

(a) if $L \not\subset P$, then there exist planes P_i such that $L, L_i \subset P_i$ for $i = 1, 2$,

(b) if $L \subset P$, then there exists a line $L_3 \not\subset P$ belonging to the bundle and a plane P_3 such that $L, L_3 \subset P_3$.

The set of all bundles will be denoted by $\overline{\boldsymbol{S}}$. If L_1 meets $L_2(\neq L_1)$ at a point p, then the bundle determined by them is the set of all lines passing through p, and conversely, to every point $p \in \boldsymbol{S}$ there corresponds exactly one bundle with this property. Thus one obtains a mapping $\psi : \boldsymbol{S} \to \overline{\boldsymbol{S}}$ called the natural injection of $\boldsymbol{S}$ into $\overline{\boldsymbol{S}}$. Any element of $\psi(\boldsymbol{S})$ or of $\complement\psi(\boldsymbol{S})$ will be termed a proper or improper point, respectively.

For any line L or any plane P in $\boldsymbol{S}$, the subset $\psi(L)$ or $\psi(P)$ in $\psi(\boldsymbol{S})$ will be called a proper line or proper plane, respectively. It will be said that an improper point lies on a proper line L or on a proper plane P, if $\psi^{-1}(L)$ belongs to its bundle or $\psi^{-1}(P)$ contains a line belonging to its bundle, respectively. A proper line L or a proper plane P with all improper points lying on it is called an extended line $\bar{L}$ or an extended plane $\bar{P}$.

The projective lines in $\overline{\boldsymbol{S}}$ are introduced as the intersections of two distinct extended planes. If a projective line contains a proper point, then it is an extended line; in the opposite case the projective line is called improper. The projective plane determined by an element $p \in \overline{\boldsymbol{S}}$ and a projective line L, $p \notin L$, is defined as the set of all elements lying on the projective lines $\boldsymbol{L}(ap)$, $a \in L$. If a projective plane contains a proper point, then it is an extended plane; in the opposite case the projective plane is termed improper. Projective lines and planes determine on $\overline{\boldsymbol{S}}$ the structure of a projective space.

Now let a_i, $i = 1, \ldots, 4$, be distinct elements in $\overline{\boldsymbol{S}}$ on the same projective line. Choose a proper point p not on this line; then the projective lines $\boldsymbol{L}(a_i p)$ are extended lines, and one may introduce the relation of division between a_i corresponding to the relation of division between the lines $\psi^{-1}(\boldsymbol{L}(a_i p))$. Now, $\overline{\boldsymbol{S}}$ is an ordered projective space, and the pair $(\overline{\boldsymbol{S}}, \psi)$ will be called the ***projective extension*** of $\boldsymbol{S}$.

From the preceding construction there follows immediately

Theorem 3. *Let* $\mathbf{S}_i$, $i = 1, 2$, *be ordered incidence spaces, and* $\varphi : \mathbf{S}_1 \to \mathbf{S}_2$ *an injection such that* $\varphi(\mathbf{S}_1)$ *is a convex set in* $\mathbf{S}_2$ *and* φ *is an isomorphism between* $\mathbf{S}_1$ *and* $\varphi(\mathbf{S}_1)$ *(see item 2). Let* $(\overline{\mathbf{S}}_i, \psi_i)$ *be projective extension of* $\mathbf{S}_i$. *Then there exists a unique isomorphism* $\bar{\varphi} : \overline{\mathbf{S}}_1 \to \overline{\mathbf{S}}_2$ *such that* $\bar{\varphi} \circ \psi_1 = \psi_2 \circ \varphi$.

5. Assume that there exists an improper plane I in $\overline{\mathbf{S}}$. On $\mathbf{A} = \overline{\mathbf{S}} \setminus I$ introduce the structure of an ordered affine space according to item 3. Since I is improper, there is $\psi(\mathbf{S}) \subseteq \mathbf{A}$. After introducing on $\psi(\mathbf{S})$ the structure of an ordered incidence space according to item 2, the mapping ψ becomes an isomorphism between $\mathbf{S}$ and $\psi(\mathbf{S})$. Indeed, it is immediate that lines and planes in $\mathbf{S}$ and $\psi(\mathbf{S})$ correspond to each other; the assertion for the betweenness relations is obtained as follows. Denote by $\overline{\mathbf{B}}$ the betweenness relation in $\psi(\mathbf{S})$; according to item 4, $\overline{\mathbf{B}}(a, b, c)$ means that for a proper point $p \notin \mathbf{L}(ab)$ the lines $\psi^{-1}(\mathbf{L}(pa))$ and $\psi^{-1}(\mathbf{L}(pc))$ divide $\psi^{-1}(\mathbf{L}(pb))$ and $\psi^{-1}(\mathbf{L}(pl_\infty))$, where $\{l_\infty\} = \mathbf{L}(ab)) \cap I$. Since l_∞ is an improper point, all points on the line $\psi^{-1}(\mathbf{L}(pl_\infty))$ lie on the half-plane determined by $\psi^{-1}(L)$ and $\psi^{-1}(p)$. In this situation it is clear that the mentioned division is satisfied if and only if $\mathbf{B}(\psi^{-1}(a), \psi^{-1}(b), \psi^{-1}(c))$. Obviously $\psi(\mathbf{S})$ is a convex set in $\mathbf{A}$. Thus

Theorem 4. *If there exists an improper plane in* $\overline{\mathbf{S}}$, *then* $\mathbf{S}$ *is isomorphic to a convex set in an affine space over an ordered skew-field.*

6. Assume that Dedekind's axiom holds in $\mathbf{S}$. Then the existence of an improper plane follows from lemmas 1–3.

Lemma 1. *Every extended line* $\bar{L}$ *contains an improper point.*

Proof. An arbitrary proper point a on the proper line L determines two half-lines L_1 and L_2 on $\psi^{-1}(L)$. Since P is a proper plane containing L, and p a proper point on P and not on L, denote by $\mathscr{P}$ the pencil on $\psi^{-1}(P)$ with vertex $\psi^{-1}(p)$ from which the line $\psi^{-1}(\mathbf{L}(ap))$ is excluded, and by $\mathscr{X}_i$ the subset in $\mathscr{P}$ formed by lines meeting L_i, $i = 1, 2$. If there were in $\mathscr{P}$ no line not meeting $\psi^{-1}(L)$, then $(\mathscr{X}_1, \mathscr{X}_2)$ would be a cut in $\mathscr{P}$ without a boundary element, because there is no "last" point on L_1 and L_2. But this is in contradiction with Dedekind's axiom.

Lemma 2. *Every improper point p lies on an improper line.*

Proof. Take an extended plane $\bar{P}$ and an extended line $\bar{L}$ such that $p \in \bar{L}$, $\bar{L} \subset \bar{P}$, $\psi^{-1}(L)$ determines two half-planes P_1 and P_2 on $\psi^{-1}(P)$. Denote by $\mathscr{P}$ the pencil of projective lines on $\bar{P}$ with vertex p from which the line $\bar{L}$ is excluded, and by $\mathscr{X}_i$ the subset in $\mathscr{P}$ formed by lines containing elements of $\psi(P_i)$, $i = 1, 2$. If there were no improper line through p, then $(\mathscr{X}_1, \mathscr{X}_2)$ would be a cut in $\mathscr{P}$ without boundary element, which is a contradiction[1]).

Lemma 3. *Every improper line lies on an improper plane.*

The proof is analogous to the preceding ones.

7. If Dedekind's axiom does not hold in **S**, then improper points need not exist, as shown by following construction. Let **Q** or **R**(**Q** $\subset$ **R**) denote the field of rational or real numbers respectively; then the projective number space **P**(**Q**) (see item 1) is a subspace in **P**(**R**). In **P**(**R**) there are planes containing no rational point, e.g. the plane $I \equiv x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 = 0$, where α is a transcendental number[2]), since from $r_0 + r_1\alpha + r_2\alpha^2 + r_3\alpha^3 = 0$, $r_i \in$ **Q** it follows that $r_i = 0$ for $i = 0, \ldots, 3$. According to item 3 we introduce on **A** = **P**(**R**) $\setminus I$ the structure of an ordered affine space. Since I contains no rational point, there is **P**(**Q**) $\subset$ **A**. On defining in **P**(**Q**) the betweenness relation by restriction from **A**, one obtains an ordered incidence space whose projective extension contains no improper point. Thus we have the interesting metageometric

Theorem 5. *The assumption* "If p is a point and L a line both on a plane P, $p \notin L$, then there exists at least one line on P passing through p and not meeting L" *cannot be deduced from Hilbert's axioms of incidence and order.*

8. From a more detailed analysis, which will be not presented here, it follows that *the ordered incidence spaces may be classified into six types* characterized by following properties of improper objects of their projective extentions:

a) there is no improper point,

b) there is exactly one improper point,

c) there is one improper line and no further improper point,

d) there are four improper points not on the same plane and no improper line,

e) there are four improper points not on the same plane, there is an improper line and no improper plane,

f) there is an improper plane.

Ordered incidence spaces of all 6 types exist.

9. In the preceding sections the use of some non-planar considerations was essential. In conclusion, we want to draw attention to the interesting problem of considering analogous projective extentions of partial planes with order axioms by means of appropriate configuration theorems.

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[1]) Notice that if Dedekind's axiom holds in **S**, then it holds also in $\bar{\mathbf{S}}$.

[2]) It evidently suffices to consider the field **Q**(α) instead **R**.

References

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Výtah

GEOMETRICKÁ STRUKTURA S HILBERTOVÝMI AXIOMY INCIDENCE A USPOŘÁDÁNÍ

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V práci se studují uspořádané incidenční prostory pomocí jejich projektivního rozšíření. V §5 se dokazuje, že každý uspořádaný incidenční prostor, v jehož projektivním rozšíření existuje nevlastní rovina, je isomorfní konvexní množině v afinním prostoru nad nějakým uspořádaným tělesem (ne nutně komutativním). V §7 je pak ukázáno, že nevlastní rovina nemusí obecně existovat. Při tom se získává i metageometrická věta 5: Tvrzení „*Nechť p a L jsou bod a přímka v rovině P, $p \notin L$, pak bodem p lze vést alespoň jednu přímku, která leží v P a neprotíná L*" nelze odvodit pouze z Hilbertových axiomů incidence a uspořádání.

Резюме

ГЕОМЕТРИЧЕСКАЯ СТРУКТУРА С АКСИОМАМИ ИНЦИДЕНТНОСТИ И ПОРЯДКА ГИЛЬБЕРТА

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В работе изучаются упорядоченные инцидентные пространства при помощи их проективного расширения. В § 5 доказано, что всякое упорядоченное инцидентое пространство, в проективном расщирении которого существует несобственная плоскость, изоморфно выпуклому множеству в аффинном пространстве над некоторым упорядоченным телом. В § 7 показано, что несобственная плоскость в общем случае не должна существовать. При этом получена метагеометрическая теорема 5: Утверждение: „*Пусть точка p и прямая L лежат в плоскости P, $p \notin L$; тогда через p проходит по крайней мере одна прямая, лежащая в P и непересекающая L*" — невозможно вывести только из аксиом инцидентности и порядка Гильберта.