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ON MINIMAX SOLUTIONS OF STOCHASTIC LINEAR PROGRAMMING PROBLEMS

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1. Our starting point is the formulation of a stochastic linear program as a strategic game. This formulation differs only slightly from that given by IOSIFESCU and THEODORESCU [3]. Secondly, we state a minimax theorem for that game and study the methods of solution. In some special but important cases it is shown that the minimax solution of a stochastic linear program is equivalent to the solution of an ordinary linear program (of greater dimension, in general). The existence of a finite solution is also discussed.

2. Let E_n^+ denote the non-negative orthant of the n -dimensional Euclidean space.

Let (A, b, c) — where $A = (a_{ij})$, $b = (b_i)$, $c = (c_j)$, $i = 1, \dots, m$, $j = 1, \dots, n$ — be a random vector; let its distribution $F(A, b, c)$ belong to a set of distributions \mathcal{F} .

Let $r_i(y)$, $i = 1, \dots, m$ be real functions such that $r_i(y) = 0$ for $y \leq 0$ and $r_i(y) > 0$ for $y > 0$.

For $x \in E_n^+$, $F \in \mathcal{F}$ set

$$H(x, F) = E_F \left\{ \sum_{j=1}^n c_j x_j - \sum_{i=1}^m r_i \left(\sum_{j=1}^n a_{ij} x_j - b_i \right) \right\}.$$

If H is defined and finite for all $x \in E_n^+$, $F \in \mathcal{F}$, define a two-person zero-sum game by its normal form

$$G = (E_n^+, \mathcal{F}, H).$$

3. The game G corresponds to the situation, where in the linear program:

$$Ax \leq b, \quad x \geq 0, \quad c'x = \text{maximum},$$

A, b, c (or some of them, or some of their components) are random vectors, their simultaneous distribution is known to belong to a set \mathcal{F} , the vector x is to be chosen

independently of the realization of these random vectors, and the violation of the constraint $\sum_{j=1}^n a_{ij}x_j \leq b_i$ is penalized by the amount

$$r_i \left(\sum_{j=1}^n a_{ij}x_j - b_i \right), \quad i = 1, \dots, m.$$

As the *solution* of this program, the optimal pure strategy of the player I in the game G will be meant.

4. For $F \in \mathcal{F}$, let F^* be the corresponding marginal distribution of (A, b) ; let $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$.

Theorem 1. *Suppose that one of the following conditions is satisfied:*

(i) *the set \mathcal{F} is convex and compact (in the sense of Lévy's distance) and the c_j 's are uniformly integrable with respect to $F \in \mathcal{F}$,*

(ii) *the set \mathcal{F}^* is convex and compact and $E_F c$ equals a constant vector for all $F \in \mathcal{F}$.*

Let the functions $r_i(y)$, $i = 1, \dots, m$, be convex and let the functions $r_i(\sum_{j=1}^n a_{ij}x_j - b_i)$ be uniformly integrable with respect to $F \in \mathcal{F}$ for every $x \in E_n^+$.¹⁾

Then we have

$$(1) \quad \sup_{x \in E_n^+} \min_{F \in \mathcal{F}} H(x, F) = \min_{F \in \mathcal{F}} \sup_{x \in E_n^+} H(x, F)$$

where this common value is either $+\infty$ or it is finite and the suprema can be replaced by maxima.

Proof. Let condition (i) be satisfied. According to FAN KY [2, Theor. 2], it is sufficient to show that H is a continuous and convex function on \mathcal{F} for each $x \in E_n^+$ and that for each element of \mathcal{F} it is a concave function on E_n^+ . The continuity of H on \mathcal{F} is a consequence of the uniform integrability of the c_j 's and r_i 's — see LOËVE [5, Theor. 11.4A] — and of the special form of H ; the convexity is trivial, because H , being an integral with respect to F , is additive and homogeneous in F . The concavity of H on E_n^+ follows easily from the convexity of the r_i 's.

Now, let condition (ii) be satisfied. For every $F \in \mathcal{F}$ let F^* be the mentioned marginal distribution; let γ be the constant vector $E_F c$. Let us define $H^*(x, F^*) =$

¹⁾ The uniform integrability of the r_i 's means that for every $x \in E_n^+$ and $i = 1, \dots, m$,

$$\lim_{N \rightarrow \infty} \int_{|a_{ij}| \geq N, j=1, \dots, n, |b_i| \geq N} \dots \int r_i \left(\sum_{j=1}^n a_{ij}x_j - b_i \right) dF = 0$$

holds uniformly in $F \in \mathcal{F}$; similarly for the c_j 's. Cf. Loève [5, p. 182].

$= \sum_{j=1}^n \gamma_j x_j - E_{F^*} \left\{ \sum_{i=1}^m r_i \left(\sum_{j=1}^n a_{ij} x_j - b_i \right) \right\}$. It is readily seen that the games $G = (E_n^+, \mathcal{F}, H)$ and $G^* = (E_n^+, \mathcal{F}^*, H^*)$ are equivalent (in the sense of BLACKWELL and GIRSHICK [1, Def. 1.4.2]) and the game G^* again satisfies the conditions of the Fan Ky's theorem.

We shall now investigate the game G for special choices of r_i and \mathcal{F} . It will be shown that in some cases the method of solution reduces to an ordinary linear program. As a rule, we shall describe the set of distributions \mathcal{F} in terms of random variables and their properties. In the sequel, Greek letters will always denote known constants (not $\pm \infty$).

5. In the game G , let a_{ij} , $i = 1, \dots, m$, $j = 1, \dots, n$, be constants, let b_i , $i = 1, \dots, m$, be independent random variables such that $E(b_i) = \beta_i$, $\beta'_i \leq b_i \leq \beta''_i$ a.s., $\beta'_i < \beta''_i$; let c_j , $j = 1, \dots, n$, be random variables such that $E c_j = \gamma_j$. (Herewith the set \mathcal{F} is defined.) Let $r_i(y) = v_i y^+$, $v_i > 0$, $i = 1, \dots, m$, where $y^+ = \frac{1}{2}(|y| + y)$.

Let $\Delta = \{I, J, K\}$ be an arbitrary decomposition of the set $\{1, 2, \dots, m\}$ into three disjoint parts, one or two of which may be empty. Denote Δ_v , $v = 1, \dots, M$, those of such decompositions, for which the sets

$$(2) \quad \{x \geq 0: \sum_{j=1}^n a_{ij} x_j \geq \beta''_i, i \in I; \beta'_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta''_i, i \in J; \sum_{i=1}^m a_{ij} x_j \leq \beta'_i, i \in K\}$$

are non-empty. ($M \leq 3^m$ holds.) Further, define

$$q_{vj} = -\gamma_j + \sum_{i \in I_v} v_i a_{ij} + \sum_{i \in J_v} v_i \lambda_i a_{ij},$$

$$k_v = \sum_{i \in I_v} v_i \beta_i + \sum_{i \in J_v} v_i \lambda_i \beta'_i \quad v = 1, \dots, M; \quad j = 1, \dots, n,$$

where $\lambda_i = (\beta''_i - \beta_i) / (\beta''_i - \beta'_i)$. Finally, denote $Q = (q_{vj})$, $v = 1, \dots, M$, $j = 1, \dots, n$; $k = (k_v)$, $v = 1, \dots, M$; $e' = (1, \dots, 1)$.

Theorem 2. (i) Relation (1) holds. (ii) x is a solution of the game G if and only if (x, y_0) is a solution of the linear program

$$(3) \quad Qx + ye \leq k, \quad x \geq 0, \quad y = \text{maximum}.$$

Proof. (i) As the b_i are independent it suffices to prove the compactness of the sets of one-dimensional distributions. But this is a consequence of the criterion: \mathcal{F} is compact $\Leftrightarrow \mathcal{F}$ is closed and both $\lim_{t \rightarrow -\infty} F(t) = 0$, $\lim_{t \rightarrow +\infty} F(t) = 1$ are uniform in \mathcal{F} .

(See Loève [5, p. 215].) All other conditions of Theorem 1 are evidently satisfied.

(ii) Denote $\varphi(x) = \min_{F \in \mathcal{F}} H(x, F)$. From the independence of b_i 's it follows

$$\varphi(x) = \sum_{j=1}^n \gamma_j x_j - \sum_{j=1}^m v_j \max_{F \in \mathcal{F}} E_F \left\{ \left(\sum_{i=1}^n a_{ij} x_j - b_i \right)^+ \right\}.$$

The terms of the second sum can be evaluated with the help of a result due to JIŘINA and NEDOMA [4, Theor. 4]¹⁾; we get

$$\varphi(x) = \sum_{j=1}^n \gamma_j x_j - \sum_{i=1}^m v_i \{ \lambda_i (\sum_{j=1}^n a_{ij} x_j - \beta'_i)^+ + (1 - \lambda_i) (\sum_{j=1}^n a_{ij} x_j - \beta''_i)^+ \}.$$

The function $\varphi(x)$ is concave, and it is linear ($\varphi(x) = -\sum_{j=1}^n q_{vj} x_j + k_v$) on each of the sets (2), which correspond to the decompositions Δ_v , $v = 1, \dots, M$; these sets themselves constitute a (non-overlapping) decomposition of E_n^+ . This means that $-\sum_{j=1}^n q_{vj} x_j + k_v$, $v = 1, \dots, M$ are the upper supporting hyperplanes to $\varphi(x)$ and that

$$\varphi(x) = \min_{1 \leq v \leq M} \left(-\sum_{j=1}^n q_{vj} x_j + k_v \right).$$

Now, the maximization of $\varphi(x)$ is equivalent to the maximization of y under the constraint $y \leq \varphi(x)$, i.e., under the constraints $y \leq -\sum_{j=1}^n q_{vj} x_j + k_v$, $v = 1, \dots, M$; but this is the assertion of the theorem.

Theorem 3. *The non-existence of a (strictly) negative column in the matrix Q is a necessary condition and the existence of a non-negative row in Q is a sufficient condition for the existence of a (finite) solution of the game G .*

Proof. The solution of the game G exists \Leftrightarrow there exists the maximum of y on the set $\{x \geq 0 : Qx + ye \leq k\}$ (which is always non-empty) \Leftrightarrow there is an admissible vector of the dual linear program to (3), i.e., there exists a vector u , for which

$$u \geq 0, \quad \sum_{v=1}^M u_v = 1, \quad \sum_{v=1}^M u_v q_{vj} \geq 0, \quad j = 1, \dots, n.$$

Especially, such a vector exists if there is a non-negative row in Q , and it cannot exist if there is a negative column in Q .

Corollary. *If the set $\{x \geq 0 : \sum_{j=1}^n a_{ij} x_j \geq \beta'_i, i = 1, \dots, m\}$ is non-empty and if $\sum_{i=1}^m v_i a_{ij} \geq \gamma_j$ for all $j = 1, \dots, n$, then a solution of the game G exists.*

Proof. To the decomposition $\{(1, \dots, m), \emptyset, \emptyset\}$ there corresponds the row in Q with non-negative components $-\gamma_j + \sum_{i=1}^m v_i a_{ij}$, $j = 1, \dots, n$.

¹⁾ The theorem holds under the assumption of convexity (concavity) only; the existence of the second derivatives is not needed.

6. If the assumption $E(b_i) = \beta_i$ is replaced by $\bar{\beta}_i \leq E(b_i) \leq \bar{\bar{\beta}}_i$, $i = 1, \dots, m$, and all the other assumption of Section 5 are unaltered, then the Theorem 2 remains true with β_i replaced by $\bar{\beta}_i$ (in the definitions of λ_i and k_v).

On the other hand, if the assumption $\beta'_i \leq b_i \leq \beta''_i$ a.s., is replaced by $\sigma(b_i) \leq \leq \text{const.}$ (say σ_i), $i = 1, \dots, m$, and all the other assumptions of Section 5 are unaltered, then part (i) of the Theorem 2 remains true – this follows from Loève [5, Theor. 11.4.A,B] – but to find a solution is much more difficult. It means first to find

$$\max_{F \in \mathcal{F}} E_F \left\{ \left(\sum_{j=1}^n a_{ij} x_j - b_i \right)^+ \right\} = \min \{ d_0 + d_1 \beta_i + d_2 (\sigma_i^2 + \beta_i^2) \},$$

where the minimum is to be taken over all (d_0, d_1, d_2) which satisfy $\left(\sum_{j=1}^n a_{ij} x_j - y \right)^+ \leq \leq d_0 + d_1 y + d_2 y^2$, $-\infty < y < +\infty$, when $x \in E_n^+$ is fixed (see RICHTER [6]), then to set this result into the relation for $\varphi(x)$ and to maximize $\varphi(x)$. (Both these comments are easily to prove.)

7. In the game G , let $(a_{i1}, \dots, a_{in}, b_i)$, $i = 1, \dots, m$ be mutually independent random vectors such that

$$(4) \quad \alpha'_{ij} \leq a_{ij} \leq \alpha''_{ij}, \quad \beta'_i \leq b_i \leq \beta''_i \quad \text{a.s.}, \quad \alpha'_{ij} < \alpha''_{ij},$$

$$E a_{ij} = \alpha_{ij} = \frac{1}{2}(\alpha'_{ij} + \alpha''_{ij}), \quad E b_i = \beta_i = \frac{1}{2}(\beta'_i + \beta''_i), \quad i = 1, \dots, m, \quad j = 1, \dots, n;$$

let c_j , $j = 1, \dots, n$ be random variables such that $E c_j = \gamma_j$. Let $r_i(y) = v_i y^+$, $v_i > 0$, $i = 1, \dots, m$. Let $\Delta_v = (I_v, J_v, K_v)$, $v = 1, \dots, M$, be all those decompositions of $\{1, \dots, m\}$, for which the sets

$$\{x \geq 0 : \sum_{j=1}^n \alpha'_{ij} x_j - \beta''_i \geq 0, \quad i \in I_v;$$

$$\sum_{j=1}^n \alpha'_{ij} x_j - \beta''_i \leq 0 \leq \sum_{j=1}^n \alpha''_{ij} x_j - \beta'_i, \quad i \in J_v; \quad \sum_{j=1}^n \alpha''_{ij} x_j - \beta'_i \leq 0, \quad i \in K_v\}$$

are non-empty. Define

$$p_{vj} = -\gamma_j + \sum_{i \in I_v} v_i \alpha_{ij} + \frac{1}{2} \sum_{i \in J_v} v_i \alpha''_{ij},$$

$$h_v = \sum_{i \in I_v} v_i \beta_i + \frac{1}{2} \sum_{i \in J_v} v_i \beta'_i$$

and denote $P = (p_{vj})$, $v = 1, \dots, M$, $j = 1, \dots, n$; $h = (h_v)$, $v = 1, \dots, M$.

Theorem 4. (i) Relation (1) holds. (ii) x is a solution of the game G if and only if (x, y_0) is a solution of the linear program

$$Px + ye \leq h, \quad x \geq 0, \quad y = \text{maximum}.$$

(Note that the case $\beta'_i = \beta''_i$ is not excluded.)

Proof. According to the theorem of Jifina - Nedoma [4]

$$\begin{aligned}
 & \max_{F \in \mathcal{F}} \mathbf{E}_F \left\{ \left(\sum_{j=1}^n a_{ij} x_j - b_i \right)^+ \right\} = \\
 & = \frac{\left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right) - \left(\sum_{j=1}^n \alpha_{ij} x_j - \beta_i \right)}{\left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right) - \left(\sum_{j=1}^n \alpha_{ij} x_j - \beta_i \right)} \left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right)^+ + \\
 & + \frac{\left(\sum_{j=1}^n \alpha_{ij} x_j - \beta_i \right) - \left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right)}{\left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right) - \left(\sum_{j=1}^n \alpha_{ij} x_j - \beta_i \right)} \left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right)^+ = \\
 & = \frac{1}{2} \left[\left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right)^+ + \left(\sum_{j=1}^n \alpha_{ij} x_j - \beta_i \right)^+ \right].
 \end{aligned}$$

The other parts of the proof are the same as in Theorem 2.

Condition (4) is essential for the linearity of resulting program only. Theorem 3 and its Corollary hold also true with P in the place of Q .

8. In the game G , let (A, b, c) be a random vector such that

$$\alpha'_{ij} \leq a_{ij} \leq \alpha''_{ij}, \beta'_i \leq b_i \leq \beta''_i, \gamma'_j \leq c_j \leq \gamma''_j \quad \text{a.s., } j = 1, \dots, n, i = 1, \dots, m.$$

Let $r_i(y) = v_i y^+$, $v_i > 0$, $i = 1, \dots, m$. Let $\Delta'_v = (I_v, J_v)$, $v = 1, \dots, M'$, be all those decompositions of $\{1, \dots, m\}$, for which the sets

$$\left\{ x \geq 0 : \sum_{j=1}^n \alpha''_{ij} x_j - \beta'_i \geq 0, i \in I_v; \sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \leq 0, i \in J_v \right\}$$

are non-empty. ($M' \leq 2^m$ holds.) Define

$$r_{vj} = -\gamma'_j + \sum_{i \in I_v} v_i \alpha''_{ij}, \quad l_v = \sum_{i \in I_v} v_i \beta'_i$$

and denote $R = (r_{vj})$, $v = 1, \dots, M'$, $j = 1, \dots, n$; $l = (l_v)$, $v = 1, \dots, M'$.

Theorem 5. (i) Relation (1) holds. (ii) x is a solution of the game G if and only if (x, y_0) is a solution of the linear program

$$Rx + ye \leq l, \quad x \geq 0, \quad y = \text{maximum}.$$

Proof. Assertion (i) follows from the fact that condition (i) of Theorem 1 is satisfied. To prove assertion (ii), let us take into account that for $x \in E_n^+$ we have

$$\min_{F \in \mathcal{F}} \mathbf{E}_F \left\{ \sum_{j=1}^n c_j x_j - \sum_{i=1}^m v_i \left(\sum_{j=1}^n a_{ij} x_j - b_i \right)^+ \right\} = \sum_{j=1}^n \gamma'_j x_j - \sum_{i=1}^m v_i \left(\sum_{j=1}^n \alpha'_{ij} x_j - \beta'_i \right)^+.$$

The rest of the proof is the same as in Theorem 2.

Theorem 3 and its Corollary also hold true with R in the place of Q .

9. It is possible to modify the above results for the case where $r_i(y) = 0$ for $y = 0$ and $r_i(y) > 0$ for $y \neq 0$, which corresponds to the relation $Ax = b$ at the place of $Ax \leq b$ in the initial linear program. Then Theorem 1 remains true; if $r_i(y) = v_i y^+ + w_i y^-$, $v_i > 0$, $w_i > 0$, $i = 1, \dots, m$, and all the other assumptions are unaltered, then Theorem 2 holds true with \tilde{Q} and \tilde{k} in the places of Q and k , where

$$\tilde{q}_{vj} = -\gamma_j + \sum_{i \in I_v} v_i a_{ij} + \sum_{i \in J_v} (v_i \lambda_i - w_i(1 - \lambda_i)) a_{ij} - \sum_{i \in K_v} w_i a_{ij},$$

$$\tilde{k}_v = \sum_{i \in I_v} v_i \beta_i + \sum_{i \in J_v} (v_i \lambda_i \beta'_i - w_i(1 - \lambda_i) \beta''_i) - \sum_{i \in K_v} w_i \beta_i,$$

and so on.

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Resumé

O MINIMAXOVÉM ŘEŠENÍ ÚLOHY STOCHASTICKÉHO LINEÁRNÍHO PROGRAMOVÁNÍ

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Obsahem článku je vyšetřování úlohy stochastického lineárního programování a jejího řešení jakožto strategické hry. V prvních paragrafech je tato úloha formulována a pro uvažovanou hru je dokázána věta o minimaxu (věta 1). V dalších paragrafech se studují metody řešení. Ve větách 2, 4 a 5 je dokázáno, že minimaxové řešení úlohy stochastického lineárního programování je v některých důležitých speciálních případech ekvivalentní řešení úlohy lineárního programování (větší dimenze). Podmínky pro existenci konečného řešení jsou uvedeny ve větě 3.

Резюме

О МИНИМАКСНОМ РЕШЕНИИ ПРОБЛЕМЫ СТОХАСТИЧЕСКОГО ЛИНЕЙНОГО ПРОГРАММИРОВАНИЯ

ЙИТКА ЖАЧКОВА (Jitka Žáčková), Прага

В настоящей статье изучается задача стохастического линейного программирования и ее решение с точки зрения теории игр. В первых отделах дана постановка задачи, и для возникшей бесконечной игры показана теорема о минимаксе (теорема 1). Остальные отделы посвящены исследованию решения задачи. Для некоторых важных частных случаев показано (теоремы 2, 4 и 5), что минимаксное решение задачи стохастического линейного программирования равносильно решению определенной задачи линейного программирования (большой размерности). В теореме 3 исследуется существование конечного решения.