Ivo Marek
On approximate methods in eigenvalue problems

Časopis pro pěstování matematiky, Vol. 92 (1967), No. 1, 89--104

Persistent URL: http://dml.cz/dmlcz/117588

Terms of use:
© Institute of Mathematics AS CR, 1967

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ON APPROXIMATE METHODS IN EIGENVALUE PROBLEMS 1)

Ivo Marek, Praha

(Received December 18, 1965)

1. DEFINITIONS AND NOTATION

Let $\mathcal{X}_1, \mathcal{X}_2$ be complex Banach spaces, $x, y, \ldots, u, v, \ldots$ its elements and $\|x\|_1, \|u\|_2$ norms of $x \in \mathcal{X}_1, u \in \mathcal{X}_2$. Let $H$ be an index set and $\mathcal{X}_{j,h}, j = 1, 2, h \in H,$ be Banach spaces with norms $\| \cdot \|_{j,h}.$ We suppose, that there exist continuous linear transformations $P_{j,h}$, $j = 1, 2, h \in H$, of $\mathcal{X}_j$ onto $\mathcal{X}_{j,h}$. Further we suppose that there exist subspaces $\mathcal{X}_{j,h}$ of $\mathcal{X}_j$ isomorphic with $\mathcal{X}_{j,h}$. Let us denote by $S_{j,h}$ the corresponding isomorphisms. Hence

$$S_{j,h}\mathcal{X}_{j,h} = \mathcal{X}_{j,h} \quad \text{or} \quad \mathcal{X}_{j,h} = S_{j,h}^{-1}\mathcal{X}_{j,h}.$$ 

We shall assume that $H = (0, h_0)$ and that for an arbitrary vector $x_j \in \mathcal{X}_j$ there holds the expression $x_j = \lim_{h \to 0} \tilde{x}_{j,h}$, where $\tilde{x}_{j,h} = S_{j,h}^{-1}P_{j,h}x_j \in \mathcal{X}_{j,h}$. Let us note, that in applications $\mathcal{X}_{j,h}$ are usually finite dimensional spaces.

Let $\mathcal{Z}$ be a complex Banach space. Then $\mathcal{Z}'$ denotes the adjoint space of continuous linear forms on $\mathcal{Z}$. The space of all bounded linear operators from $\mathcal{Z}$ into the Banach space $\mathcal{Z}_1$, topologized by the norm $\|T\| = \sup_{\|x\|_\mathcal{Z}=1} \|Tx\|_{\mathcal{Z}_1}, x \in \mathcal{Z},$ $Tx \in \mathcal{Z}_1$, will be denoted by $[\mathcal{Z}, \mathcal{Z}_1]$. We put $[\mathcal{Z}] = [\mathcal{Z}, \mathcal{Z}].$

Let $T \in [\mathcal{Z}, \mathcal{Z}_1]$. Then $T'$ denotes the adjoint operator, i.e. the operator $T'$, for which $y' = T'x' \iff y'(x) = x'(Ty)$, where $y = Tx$ and $x \in \mathcal{Z}, \ y \in \mathcal{Z}_1, \ x' \in \mathcal{Z}', \ y' \in \mathcal{Z}'$. Evidently $T' \in [\mathcal{Z}', \mathcal{Z}']$.

If $T$ is a linear (not necessarily) bounded transformation, then $\mathcal{R}(T), \mathcal{R}(T) \subset \mathcal{Z}$ denotes the domain and $\mathcal{R}(T), \mathcal{R}(T) \subset \mathcal{Z}_1$ the range of this transformation.

The following eigenvalue problem

$$\tag{1.1} Tx = \mu x$$

shall be investigated together with the "approximate" eigenvalue problem

\[ T_h x_h = \mu^{(h)} x_h , \]

where \( T, T_h \) are elements of \( \mathcal{X}, \mathcal{X}_h \) and the symbols \( \mathcal{X}, \mathcal{X}_h \) denote the corresponding pair of the spaces \( \mathcal{X}_j, \mathcal{X}_{j,h}, j = 1, 2 \). In agreement with this notation we also omit the index \( j, j = 1, 2 \), in \( P_{j,h} \).

The problem (1.2) can be considered as an approximation of the problem (1.1) only if the closeness of the operators \( T \) and \( T_h \) is guaranteed. This closeness we shall define later. Then the eigenvalue problem (1.2) will be called the approximate problem of (1.1) and the eigenvector \( x_h \) of \( T_h \) an approximate eigenvector corresponding to the approximate eigenvalue \( \mu^{(h)} \). The closeness of the operators mentioned will be defined by using the terms usual in the theory of approximate methods, in particular in the net method of approximate solution of differential equations.

**Convention.** The positive constants independent of \( h \in H \) will be denoted by the unique symbol \( c \) without the distinguishing indices.

**Definition 1.** Suppose that \( T \) maps \( \mathcal{D}(T) \subset \mathcal{X}_1 \) into \( \mathcal{X}_2 \) and \( T_h \) maps \( \mathcal{D}(T_h) \subset \mathcal{X}_{1,h} \) into \( \mathcal{X}_{2,h} \). Let \( \mathcal{M}_1 \subset \mathcal{X}_1 \) and \( \mathcal{M}_1 \cap \mathcal{D}(T) = \mathcal{M} \neq \emptyset \). Let \( P_{1,h} \mathcal{M} = \mathcal{M}_h \subset \mathcal{D}(T_h) \).

Finally let \( r \) be a positive integer. If the inequalities

\[ \| P_{2,h} T x - T_h P_1 h x \|_2 \leq c(x) h^r \]

hold for all vectors \( x \in \mathcal{M} \), we shall say, that \( T_h \) have the approximation order \( r \) on the set \( \mathcal{M} \subset \mathcal{X} \) with respect to the operator \( T \).

Note, that the approximation order \( r \) depends on \( \mathcal{M}, \mathcal{X}_2, \mathcal{X}_{1,h}, \mathcal{X}_{2,h} \) and \( T \).

Let \( y \in \mathcal{X}_2 \) and put \( y_h = P_{2,h} y \); thus \( y_h \in P_{2,h} \mathcal{X}_2 \). Let \( u, u^{(h)} \) be solutions of the equations

\[ T x = y, \quad T_h x^{(h)} = y_h, \]

where the operators \( T \) and \( T_h \) map \( \mathcal{D}(T) \subset \mathcal{X}_1 \) into \( \mathcal{X}_2 \) and \( \mathcal{D}(T_h) \subset \mathcal{X}_{1,h} \) into \( \mathcal{X}_{2,h} \) respectively.

**Definition 2.** Let the equations (1.4) have unique solutions \( u \in \mathcal{D}(T), u^{(h)} \in \mathcal{D}(T_h) \) for given \( y \in \mathcal{M}_2 \subset \mathcal{X}_2 \). Let \( p \) be a positive integer. We say, that \( T_h \) has the accuracy-order \( p \) on the set \( \mathcal{M}_2 \) with respect to the operator \( T \), if there holds the inequality

\[ \| P_{1,h} u - u^{(h)} \| \leq c(y) h^p, \]

where \( c(y) \leq c \| y \|_2 \).

Evidently the accuracy-order depends on \( \mathcal{X}_1, \mathcal{M}_2, \mathcal{X}_{1,h}, \mathcal{X}_{2,h} \) and \( T \).

In definitions 1 and 2 there are given two significant characteristics of an approximate operator with respect to the investigated operator \( T \). There are some relations...
between the approximation-order \( r \) and the accuracy-order \( p \) if the operator \( T \) has, say, some smoothness properties. For example, in the net method for the differential operator \( L \), where

\[
Lu = -\frac{d}{dt} \left[ k(t) \frac{du}{dt} \right] + q(t) u(t), \quad u(0) = \alpha, \quad u(1) = \beta,
\]

in the space \( \mathcal{C}^p(\langle 0, 1 \rangle) \) of functions continuous with their derivatives of order \( p \) the accuracy-order \( p \) is equal to the approximation-order \( r \), if \( k \in \mathcal{C}^{p-1}(\langle 0, 1 \rangle) \) and \( q \in \mathcal{C}^{p-2}(\langle 0, 1 \rangle) \) (see [5]). On the other hand, there is shown in [5] that a fixed operator \( L_h \) in \( \mathcal{X}_h \) may be an approximate operator with respect to two different operators \( L_1, L_2 \) in \( \mathcal{X} \), where the corresponding approximation-orders are identical in a class of sufficiently smooth coefficients. For example if the coefficients of \( L_1 \) are smooth and the coefficients of \( L_2 \) are discontinuous, then for the accuracy-orders the inequality \( p_1 \leq p_2 \) is true, however \( p_1 \neq p_2 \) in general.

In definition 2 the equations were supposed to fulfil the unicity conditions. This definition is not convenient for the studying of eigenvalue problems.

**Definition 3.** Let \( \mu_0 \) be an eigenvalue of the operator \( T \) from \( \mathcal{D}(T) \subset \mathcal{X} \) into \( \mathcal{X} \) and let \( x_1, \ldots, x_s \), \( 1 \leq s \leq +\infty \), be corresponding linear independent eigenvectors. We say, that an operator \( T_h \) from \( \mathcal{D}(T_h) \subset \mathcal{X}_h \) into \( \mathcal{X}_h \) has the accuracy-order \( p \) in \( \mathcal{X} \) for the proper value \( \mu_0 \) with respect to the operator \( T \), if for every proper vector \( x_j \) there exist proper values \( \mu_j = 1, \ldots, l_j \) of the operator \( T_h \) and corresponding proper vectors \( x_j^{(h)} \) such, that the inequalities

\[
\left\| P_h x_j - \sum_{t=1}^{l_j} x_j^{(h)} \right\| \leq c(x_j) h^p, \quad c(x_j) \leq c \| x_j \|,
\]

hold for appropriate complex numbers \( \alpha_j \) and positive integers \( l_j \).

**Convention.** Let \( M \) and \( C \) be linear transformations from \( \mathcal{D}(M) \subset \mathcal{X}_1 \) and \( \mathcal{D}(C) \subset \mathcal{X}_1 \) into \( \mathcal{X}_2 \) and similarly \( M_h \) and \( C_h \) from \( \mathcal{D}(M_h) \subset \mathcal{X}_{1h} \) and \( \mathcal{D}(C_h) \subset \mathcal{X}_{1h} \) into \( \mathcal{X}_{2h} \). The pair \( \{M_h, C_h\} \) will be called an approximate scheme for \( M \) and \( C \), shortly a scheme.

**Definition 4.** Suppose that the equations

\[
M x = C v, \quad M_h x^{(h)} = C_h P_{1,h} v
\]

have unique solutions \( u \in \mathcal{D}(M) \) and \( u^{(h)} \in \mathcal{D}(M_h) \). We say, that the scheme \( \{M_h, C_h\} \) has the accuracy-order \( p \) in \( \mathcal{X}_1 \) with respect to the problem (1.7), if there hold the inequality

\[
\left\| P_{1h} u - u^{(h)} \right\| \leq c(v) h^p, \quad c(v) \leq c \| v \|.
\]
Definition 5. Suppose that there exist the inverse operators $M^{-1}$ and $M_h^{-1}$ and that $\mathcal{R}(M^{-1}) \subset \mathcal{D}(C)$ and $\mathcal{R}(M_h^{-1}) \subset \mathcal{D}(C_h)$, where $M, M_h, C, C_h$ are linear operators mapping $\mathcal{D}(M) \subset X_1$ and $\mathcal{D}(C) \subset X_1$ into $X_2$ and $\mathcal{D}(M_h) \subset X_{1,h}$ and $\mathcal{D}(C_h) \subset X_{1,h}$ into $X_{2,h}$ respectively. We say that the schema $\{M, C_h\}$ has the accuracy-order $p$ for the characteristic value $\lambda_0$ of the problem

\[(1.9) \quad Mx = \lambda Cx, \]

if $M_h^{-1}C_h$ or $C_hM_h^{-1}$ has the accuracy-order $p$ for the proper value $\mu_0 = \lambda_0^{-1}$ with respect to the corresponding operator $M^{-1}C$ in $X_1$ or $CM^{-1}$ in $X_2$.

Convention. We write $\{M_h, C_h; \lambda_0\}$, if we want emphasize the fact, that the scheme $\{M_h, C_h\}$ is used for the construction of the characteristic value $\lambda_0$.

2. EIGENVALUE PROBLEMS

In this paragraph we introduce some assumptions and properties of operators $M$ and $C$ which work in our eigenvalue problem

\[(2.1) \quad Mx = \lambda Cx. \]

It is assumed that $M$ and $C$ are linear, generally unbounded, operators from $\mathcal{D}(M) \subset X_1$ and $\mathcal{D}(C) \subset X_1$ into $X_2$. Further it is assumed that $\mathcal{D}(M)$ and $\mathcal{D}(C)$ are dense in $X_1$.

Together with the problem (2.1) we shall investigate the problem

\[(2.2) \quad M_h\chi^{(h)} = \lambda^{(h)}C_h\chi^{(h)}, \]

where $M_h$ and $C_h$ fulfil the same conditions.

Our purpose is to determine the accuracy-order of the scheme $\{M_h, C_h; \lambda\}$ for the problem (2.1) under the assumption that the accuracy-order of the scheme $\{M_h, C_h\}$ in the sense of definition 4 is known.

In order to exploit this assumption the eigenvalue problems (2.1) and (2.2) need be transferred into the unhomogeneous problems of type

\[(2.3) \quad Mx = y, \quad M_h\chi^{(h)} = y_h. \]

Such procedure is suitable especially for that reason that using it, one can omit some assumptions which are usually laid on $M$ and $C$ and which are as a rule too restrictive. For example the symmetry or positive definiteness. This method has also practical use. Using, e.g., Kellogg's iterations, the original eigenvalue problem is transferred into the system of countable unhomogeneous equations. As solution of this system of unhomogeneous equations the exact proper elements can be obtained.
If we restrict the infinite system to finite number of equations, we obtain some approximate proper elements. The method just described is often used for solution of time-independent problems of reactor physics [1], [3].

Though on this place the iterative process mentioned only forms an auxiliary apparatus we shall demonstrate it here in such a form which it applies directly in practical problems.

At first, we introduce some properties of the investigated operators and using these properties we demonstrate some relations needed.

Suppose \( T \in \mathcal{F} \), where \( \mathcal{F} \) is some Banach space, has a dominant eigenvalue \( \mu_0 \), i.e. \( \mu_0 \in \sigma(T) \) and the inequality

\[
|\lambda| < |\mu_0|
\]

hold for every \( \lambda \in \sigma(T) \), \( \lambda \neq \mu_0 \).

Let the symbol \( I \) denote the unity operator in \( \mathcal{F} \) and \( R(\lambda, T) = (\lambda I - T)^{-1} \) the resolvent of \( T \). Let \( \varrho_0 \) be such that for \( K \cap \sigma(T) = \{ \mu_0 \} \) holds for \( K = \{ |\lambda| |\lambda - \mu_0| \leq \varrho_0 \} \). Let \( C_0 = \{ |\lambda| \lambda - \mu_0| = \varrho_0 \} \) and

\[
B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T) \, d\lambda.
\]

It is known ([4] p. 306) that the operators \( B_1, B_{k+1} = (T - \mu_0 I) B_k, k = 1, 2, \ldots \) form the main part of Laurent expansion of the resolvent \( R(\lambda, T) \) in a neighbourhood of the singularity \( \mu_0 \). Thus

\[
R(\lambda, T) = \sum_{k=0}^{\infty} A_k(\lambda - \mu_0)^k + \sum_{k=1}^{\infty} B_k(\lambda - \mu_0)^{-k},
\]

where \( A_k \in \mathcal{F} \).

Suppose that \( x', y', x'_n, y'_n, z'_n \in \mathcal{F}' \) and that for every vector \( x \in \mathcal{F} \) there hold

\[
\lim_{n \to \infty} x'_n(x) = x'(x), \quad \lim_{n \to \infty} y'_n(x) = y'(x). \quad \lim_{n \to \infty} z'_n(x) = y'(x).
\]

Finally let there exists a positive integer \( s \), \( 1 \leq s < +\infty \) such that

\[
x'(B_s x^{(0)}) \neq 0, \quad y'(B_s x^{(0)}) \neq 0, \quad B_{s+1} x^{(0)} = 0,
\]

where \( x^{(0)} \in \mathcal{F} \) is a suitable element.

**Theorem** ([2]) Let \( \mu_0 \) be a dominant proper value of the operator \( T \in \mathcal{F} \) and let the conditions (2.6) and (2.7) be fulfilled. Then \( \lim_{n \to \infty} ||x_n - x_0|| = 0, \lim_{n \to \infty} \lambda_n = \mu_0^-1 \), where \( x_0 = [x'(B_s x^{(0)})]^{-1} B_s x^{(0)} = \mu_0^{-1} T x_0 \) and where

\[
x_n = T^n x_{(n)}, \quad x_{(n)} = \frac{x_n}{x'_n(T^n x_{(n)})}, \quad \lambda_n = \frac{y'_n(T^n x_{(n)})}{z'_n(T^{n+1} x_{(n)})}.
\]
If, moreover, \( \mu_0 \) is a pole of the resolvent of order \( q \), then we have

\[
\lim_{n \to \infty} \left| n^{-q+1} \mu_0^{-n} T^n - \frac{\mu_0^{-q+1}}{(q-1)!} B_q \right| = 0
\]

and in particular for \( q = 1 \)

\[
\lim_{n \to \infty} \mu_0^{-n} T^n = B_1.
\]

In [2] it was shown how the iterative process (2.8) and (2.9) need be modified for problems with unbounded operators. This modification will be used for transferring the problems (2.1) and (2.2) into systems of uninhomogeneous equations.

Suppose, that there exist the bounded inverses \( M_l^{-1} \) and \( M_h^{-1} \) that \( M_l^{-1} \in [\mathcal{X}_2, \mathcal{X}_1] \) and \( M_h^{-1} \in [\mathcal{X}_2, \mathcal{X}_1] \). Further let \( \mathcal{D}(C) \supset \mathcal{R}(M_l^{-1}) \) and \( \mathcal{D}(C_h) \supset \mathcal{R}(M_h^{-1}) \). Then the problems (2.1) and (2.2) are equivalent with the problems

\[
x = \lambda M_l^{-1} C x, \quad x^{(h)} = \lambda^{(h)} M_h^{-1} C_h x^{(h)}
\]

and (putting \( y = C x \) and \( y_h = C_h x^{(h)} \)) also with the problems

\[
y = \lambda C M_l^{-1} y, \quad y_h = \lambda^{(h)} C_h M_h^{-1} y_h.
\]

If \( C \in [\mathcal{X}_1, \mathcal{X}_2] \) and \( C_h \in [\mathcal{X}_1, \mathcal{X}_2] \), then \( M_l^{-1} C \in [\mathcal{X}_1] \) and \( M_h^{-1} C_h \in [\mathcal{X}_1,h] \). Corresponding iterative processes are then defined as follows

\[
M u^{(n+1)} = C u^{(n)}, \quad u_{(n+1)} = \lambda^{(n)} u^{(n+1)}, \quad u^{(0)} = x^{(0)},
\]

\[
\lambda^{(n)} = \frac{y^{(n)}(u^{(n)})}{z^{(n)}(u^{(n+1)})};
\]

\[
M_h u_h^{(n+1)} = C_h u_h^{(n)}, \quad u_h^{(n+1)} = \lambda_h^{(n)} u_h^{(n+1)}, \quad u_h^{(0)} = P_{1,h} x^{(0)},
\]

\[
\lambda_h^{(n)} = \frac{y^{(h)}(u_h^{(n)})}{z^{(h)}(u_h^{(n+1)})}.
\]

If \( CM_l^{-1} \in [\mathcal{X}_2] \) and \( C_h M_h^{-1} \in [\mathcal{X}_2,h] \), the iterative processes are defined by formulae

\[
M v^{(n)} = v^{(n)}, \quad v_{(n+1)} = v^{(n)} C u^{(n)}, \quad v^{(0)} = C x^{(0)},
\]

\[
v^{(n)} = \frac{y^{(n)}(v^{(n)})}{z^{(n)}(C v^{(n)})};
\]

\[
M_h v_h^{(n)} = v_h^{(n)}, \quad v_h^{(n+1)} = v_h^{(n)} C_h u_h^{(n)}, \quad v_h^{(0)} = C_h P_{1,h} x^{(0)},
\]

\[
v_h^{(n)} = \frac{y^{(h)}(v_h^{(n)})}{z^{(h)}(C_h v_h^{(n)})}.
\]
It is easy to see that using iterations (2.12) and (2.13) or (2.14) and (2.15) respectively, the problems (2.1) and (2.2) are transferred into the systems of uninhomogeneous equations of type

\[ Mu^{(n)} = v_n, \quad M_hu^{(h)} = v^{(h)}_n. \]

The procedure just described forms the base of the method of dehomogenization. In the theoretical considerations an appropriate choice of the initial element of iterations enables us to derive some estimates needed.

3. AUXILIARY ASSERTIONS

Suppose \( T \in [\mathcal{X}] \) and \( T_h \in [\mathcal{X}_h] \). Define the operators

\[ S = \mu_0^{-1}T, \quad S_h = \mu_0^{-1}T_h, \]

where \( \mu_0 \) and \( \mu_{0,h} \) are dominant proper values of the operators \( T \) and \( T_h \). Moreover, assume that \( \mu_0 \) and \( \mu_{0,h} \) be simple poles of the resolvents \( R(\lambda, T) \) and \( R(\lambda, T_h) \). Evidently \( S \in [\mathcal{X}] \) and \( S_h \in [\mathcal{X}_h] \) and for the spectral radii \( r(S) \) and \( r(S_h) \) we have

\[ r(S) = 1, \quad r(S_h) = 1. \]

We shall consider the operators

\[ Q_n = \sum_{k=0}^{n} S^k_h, \quad n = 1, 2, \ldots \]

From the assumptions about the dominantness of poles \( \mu_0, \mu_{0,h} \) there hold the relations \( \mathcal{C}_0 \cap \sigma(S) = \{1\}, \quad \mathcal{C}_{0,h} \cap \sigma(S_h) = \{1\} \), where \( \mathcal{C}_0 = \mu_0^{-1}C_0 \) and \( \mathcal{C}_{0,h} = \mu_0^{-1}C_{0,h} \). Let \( C_{1,h} = \{ |\lambda| = \varepsilon_1 \} \), \( K_{1,h} = \{ |\lambda| \leq \varepsilon_1 \} \) and \( K_{1,h} \cap \sigma(S_h) = \sigma(S_h) - \{1\} \). Hence \( \varepsilon_1 < 1 \).

Let us put

\[ V_n = \frac{1}{2\pi i} \sum_{k=0}^{n} \int_{\mathcal{C}_{0,h}} \lambda^k R(\lambda, S_h) \, d\lambda, \]

\[ W_n = \frac{1}{2\pi i} \sum_{k=0}^{n} \int_{\mathcal{C}_{1,h}} \lambda^k R(\lambda, S_h) \, d\lambda. \]

Thus one can write \( Q_n = V_n + W_n \).

**Lemma 3.1.** There exists a constant \( c \) independent of \( n \) such that

\[ \| W_n \| \leq c. \]

**Proof.** Easily one can see, that

\[ W_n = \frac{1}{2\pi i} \int_{\mathcal{C}_{1,h}} \frac{1 - \lambda^n}{1 - \lambda} R(\lambda, S_h) \, d\lambda. \]
According to the inequality \( q_1 < 1 \) there follows the estimate

\[
\| W_n \| \leq \frac{2q_1}{1 - q_1} \sup_{x \in C_{1,h}} \| R(\lambda, S_0) \| = c.
\]

**Lemma 3.2.** Suppose \( \mu_{0,h} \) is a simple pole of the resolvent \( R(\lambda, T_h) \). Then we have

\[
V_n = (n + 1) B_{1,h},
\]

where

\[
B_{1,h} = \frac{1}{2\pi i} \int_{C_{0,h}} R(\lambda, T_h) d\lambda.
\]

**Corollary 1.** If \( u_0 = \mu_0^{-1} T u_0 \), then we have

\[
v_n^{(h)} = V_n (P_h S u_0 - S_h P_h u_0) = o.
\]

Evidently this relation is non trivial especially, if

\[
B_{1,h} P_h u_0 = o.
\]

**Proof.** The relation (3.7) follows from the Cauchy theorem as a consequence the simplicity of the pole \( \mu_{0,h} \).

Further we shall prove the validity of (3.8). Since \( u_0 = S u_0 \), we have

\[
v_n^{(h)} = (n + 1) B_{1,h} (P_h u_0 - S_h P_h u_0).
\]

On the other hand \( B_{1,h} S_h = S_h B_{1,h} \). In the case \( B_{1,h} P_h u_0 = o \) the equation (3.8) is trivial. In the case \( B_{1,h} P_h u_0 = o \) the vector \( u_n^{(h)} = B_{1,h} P_h u_0 \) is a proper vector of \( S_h \) corresponding to the proper value 1 and thus

\[
B_{1,h} S_h P_h u_0 = S_h B_{1,h} P_h u_0 = B_{1,h} P_h u_0.
\]

From those relations (3.8) follows immediately.

Note that one can write (3.8) explicitly as follows

\[
B_{1,h} (\mu_0^{-1} P_h T u_0 - \mu_{0,h}^{-1} T_h P_h u_0) = o.
\]

**Corollary 2.** Assume

\[
\| \mu_0^{-1} P_h T u_0 - \mu_{0,h}^{-1} T_h P_h u_0 \| \leq c(u_0) h^p,
\]

where \( u_0 = \mu_0^{-1} T u_0 \), \( u_0 = o \), and \( c(u_0) \) does not depend on \( h \). Then the estimate

\[
\lim_{n \to \infty} \| \mu_0^{-n} P_h T^n u_0 - \mu_{0,h}^{-n} T_h^n P_h u_0 \| \leq c h^p
\]

is true with a constant \( c \) independent of \( h \).
Proof. It is easy to see that
\[\mu_0^{-\alpha} P_h T_h^{n+1} u_0 - \mu_0^{-\alpha} T_h^{n+1} P_h u_0 = P_h u_0 - S_h^{n+1} P_h u_0 = \sum_{k=0}^{n} [S_h^k - S_h^{k+1}] P_h u_0 = \sum_{k=0}^{n} S_h^k (P_h S - S_h P_h) u_0.\]

Lemmae (3.1) and 3.2 imply the inequalities
\[\| \sum_{k=0}^{n} S_h^k (P_h S - S_h P_h) u_0 \| = \| (V_n + W_n) (P_h S - S_h P_h) u_0 \| \leq c \| P_h S u_0 - S_h P_h u_0 \| \leq \chi^p.\]

From this we easily obtain the estimate wanted.

**Theorem 3.1. Assumptions:**

(a) Operators $T \in [X]$ and $T_h \in [X_h]$ have dominant proper values $\mu_0$ and $\mu_0, h$ and these values are simple poles of resolvents $R(\lambda, T)$ and $R(\lambda, T_h)$. Let $u_0 \in X$ be a proper vector of the operator $T$ corresponding to $\mu_0$. Then we put $u^{(h)}_0 = B_{1, h} P_h u_0$.

(b) There exists such a constant $c$ that
\[\| T_h \| \leq c.\]

(c) There exists such a constant $c$ that the estimate
\[(3.13) \quad \| P_h T u_0 - T_h P_h u_0 \| \leq \chi^p\]
holds for $h \in H$.

(d) There exist linear forms $\chi'_h \in X'_h, h \in H$, with the following properties:

(i) There exists a constant $c$ such that
\[(3.14) \quad \| \chi'_h \| \geq c.\]

(ii) The relation
\[(3.15) \quad \chi'_h (T_h x_h) = \mu_0, h \chi'_h (x_h)\]
holds for arbitrary vector $x_h \in X_h$.

(iii) There exists a constant $c$ such that
\[(3.16) \quad \| \chi'_h (P_h u_0) \| \geq c > 0.\]

Then the following assertions are valid:
There exists a constant $c$ such that

\[(3.17) \quad \| \mu_0 - \mu_{0, h} \| \leq \chi^p,\]
\[(3.18) \quad \| P_h u_0 - u^{(h)}_0 \| \leq \chi^p.\]
Proof. At first we shall prove the inequality (3.17). For this purpose we shall derive an equation for the quantity \( A_i = i^{*} - \frac{1}{\mu_0} \). We introduce the vector \( z_0^{(h)} = u_0^{(h)} - P_h u_0 \) and substitute \( P_h u_0 + z_0^{(h)} \) for \( u_0^{(h)} \) in the equation \( u_0^{(h)} = \mu_0^{-1} T_h u_0^{(h)} \). We obtain
\[
z_0^{(h)} + P_h u_0 = \mu_0^{-1} T_h P_h u_0 + \mu_0^{-1} T_h z_0^{(h)}
\]
or equivalently
\[
z_0^{(h)} - \mu_0^{-1} T_h z_0^{(h)} = w_h,
\]
where
\[
w_h = -P_h u_0 + \mu_0^{-1} T_h P_h u_0 = \mu_0^{-1}(T_h P_h - P_h T_u_0) - (\mu_0^{-1} - \mu_0^{-1}) T_h P_h u_0.
\]
By assumption \( \mu_0, h \) is a simple pole of \( R(\lambda, T_h) \) and hence the equation (3.20) has a solution iff
\[
u'(w_h) = 0
\]
holds for every form \( u' \in \mathcal{X}_h \) for which
\[
u'(x_h) = \mu_0^{-1} u'(T_h x_h)
\]
for \( x_h \in \mathcal{X}_h \). In particular the relation (3.22) must be fulfilled for the form \( \mathcal{X}_h \) having properties (di) to (diii). From the equation (3.21) we get the expression for \( \Delta \mu \) as follows
\[
\Delta \mu = \frac{\mathcal{X}'(T_h P_h u_0 - P_h T_u_0)}{\mu_0 \mathcal{X}'(T_h P_h u_0 - P_h T_u_0)} = \frac{\mathcal{X}'(T_h P_h u_0 - P_h T_u_0)}{\mu_0 \mathcal{X}'(P_h u_0)}.
\]
Thus there exists a constant \( c \) such that the estimate (3.17) is true.

Using (3.17) and (3.13) the inequality (3.11) can be easily proved and this inequality implies (3.12) as a consequence of corollary 2 of lemma 3.2. The estimate (3.12) can then be written as \( \|P_h u_0 - B_{1,h} P_h u_0\| \leq c h^p \) and this is the inequality (3.18) which was to be proved.

4. ACCURACY-ORDER AND APPROXIMATE METHODS IN EIGENVALUE PROBLEMS

In this paragraph we shall investigate the accuracy-order of an approximate scheme \( \{M_h, C_h ; \lambda \} \) under the assumption that the accuracy-order of the scheme \( \{M_h, C_h \} \) with respect to the unhomogeneous problem \( M u = C v \) is known.

Theorem 4.1. Assumptions:
1. \( C \in [\mathcal{X}_1, \mathcal{X}_2], C_h \in [\mathcal{X}_{1,h}, \mathcal{X}_{2,h}] \).
2. The operators $M$ and $M_h$ from $\mathcal{D}(M) \subset \mathcal{X}_1$ and $\mathcal{D}(M_h) \subset \mathcal{X}_{1,h}$ into $\mathcal{X}_2$ and $\mathcal{X}_{2,h}$ respectively have inverses $M^{-1}$ and $M_h^{-1}$ such that $M^{-1}C \in [\mathcal{X}_1]$ and $M_h^{-1}C_h \in [\mathcal{X}_{1,h}]$.

3. The operators $T = M^{-1}C$ and $T_h = M_h^{-1}C_h$ have dominant proper values $\mu_0$ and $\mu_{0,h}$ and these values are simple poles of $R(\lambda, T)$ and $R(\lambda, T_h)$.

We put $u_0^{(h)} = B_{1,h}P_{1,h}u_0$, if $u_0$ is a proper vector of $T$ corresponding to $\mu_0$.

4. The operators $T_h$, $h \in H$, are uniformly bounded:

$$\|T_h\| \leq c.$$

5. For every $h \in H$ there is a form $\hat{x}_h' \in \mathcal{X}_{1,h}'$ such that

(i) $$\|\hat{x}_h'\| \leq c,$$

(ii) $$\hat{x}_h'(T_hx_h) = \mu_{0,h}\hat{x}_h'(x_h) \text{ for } x_h \in \mathcal{X}_h,$$

(iii) there exists a constant $c$ such that

$$|\hat{x}_h'(P_nu_0)| \geq c > 0.$$

6. The scheme $\{M_h, C_h\}$ has accuracy-order $p$ with respect to problem (1.7) in $\mathcal{X}_1$.

**Assertion.** The scheme $\{M_h, C_h; \lambda_0\}$ has the accuracy-order $p$ for the characteristic value $\lambda_0$ with minimal modulus of the problem $Mu = \lambda Cu$.

**Proof.** Assumptions 1 to 5 guarantee the fulfilment of assumptions (a), (b) and (d) of theorem 3.1. Thus it is sufficient to legalize only assumption (c) of theorem 3.1.

Let us put $v = u_0$, where $Mu_0 = \lambda_0 Cu_0$, $\lambda = \mu_0^{-1}$, $u_0 \neq 0$. Then the equations

$$Mu = Cv, \quad M_hu_0^{(h)} = C_hP_nv$$

have unique solutions $Tu_0$ and $T_hP_{1,h}u_0$. From assumption 6 it follows that

$$\|P_hTu_0 - T_hP_nu_0\| \leq ch^p.$$

But this is an inequality required in (c) of theorem 3.1. The assertion of theorem 4.1 is then a direct consequence of theorem 3.1.

If $C$ is an unbounded operator the situation is more complicated.

**Theorem 4.2.** Assumptions:

1. The operators $M$ and $M_h$ from $\mathcal{D}(M) \subset \mathcal{X}_1$ and $\mathcal{D}(M_h) \subset \mathcal{X}_{1,h}$ into $\mathcal{X}_2$ and $\mathcal{X}_{2,h}$ have inverses $M^{-1} \in [\mathcal{X}_2, \mathcal{X}_1]$ and $M_h^{-1} \in [\mathcal{X}_{2,h}, \mathcal{X}_{1,h}]$.

2. $\mathcal{R}(M^{-1}) \subset \mathcal{D}(C)$ and $\mathcal{R}(M_h^{-1}) \subset \mathcal{D}(C_h)$.
3. There exist a constant $c$ such that

$$
\|Cy\|_2 \leq c\|y\|_1
$$

for every $y \in \mathcal{Y}(M^{-1})$ and

$$
\|C_hy_h\|_2 \leq c\|y_h\|_1
$$

for every $y_h \in \mathcal{Y}(M_h^{-1})$.

4. For every $h \in H$

$$
P_{1,h}M^{-1}\mathcal{X}_2 \subset M_h^{-1}\mathcal{X}_{2,h}.
$$

5. The operators $CM^{-1} \in [\mathcal{X}_2]$ and $C_hM_h^{-1} \in [\mathcal{X}_{2,h}]$ have dominant proper values $\mu_0$ and $\mu_{0,h}$, and these values are simple poles of resolvents $R(\lambda, CM^{-1})$ and $R(\lambda, C_hM_h^{-1})$.

If $v_0$ is a proper vector of $CM^{-1}$ corresponding to $\mu_0$, we put $v^{(h)}_0 = B_{1,h}^{(2)}P_{2,h}v_0$, where

$$
B_{1,h}^{(2)} = \frac{1}{2\pi i} \int_{C_{h,0}} R(\lambda, C_hM_h^{-1}) \, d\lambda
$$

and

$$
C_{h,0} = \{\lambda \mid |\lambda - \mu_{0,h}| \leq \varrho_{h,0}\}, \quad K_{h,0} = \{\lambda \mid |\lambda - \mu_0| \leq \varrho_{h,0}\},
$$

$$
K_{h,0} \cap \sigma(C_hM_h^{-1}) = \{\mu_{0,h}\}.
$$

6. For every $u \in \mathcal{Y}(M^{-1})$ there exists a vector $y_h \in \mathcal{X}_{2,h}$ such that

$$
C_hP_{1,h}u = P_{2,h}Cu + y_h,
$$

where $\|y_h\| \leq ch^p$. In other words, the approximation-order of $C_h$ is equal $p$ with respect to $C$ in $\mathcal{Y}(M^{-1})$.

7. The operators $C_hM_h^{-1}$, $h \in H$, are uniformly bounded: $\|C_hM_h^{-1}\| \leq c$.

8. There are forms $\mathcal{X}_2' \in \mathcal{X}_{2,h}$, $h \in H$, for which relations (i) to (iii) of assumptions of theorem 4.1 are valid, where $T_h = C_hM_h^{-1}$.

9. The scheme $\{M_h, I_h\}$ has the accuracy order $p$ in $\mathcal{X}_2$ with respect to the problem $Mu = v$, $v \in \mathcal{X}_2$.

Assertion. The scheme $\{M_h, C_h\}$ has the accuracy-order $p$ for the characteristic value $\lambda_0$ with minimal modulus of the problem $Mu = \lambda Cu$.

Proof. Similarly as in the proof of theorem 4.1 it is sufficient to legalize the fulfillment of the assumption (c) of theorem 3.1, since the other assumptions of this theorem are fulfilled as a consequence of assumptions of theorem 4.2.

We shall consider vectors $P_hCM^{-1}v_0 - C_hM_h^{-1}P_{2,h}v_0$, $h \in H$, where $v_0 = \lambda_0CM^{-1}v_0$, $\lambda_0 = \mu_0^{-1}$.  

100
The assumption 9 implies the existence of a constant c such that $\|w_h\| \leq ch^p$, where $w_h = P_1M^{-1}v_0 - M_h^{-1}P_2v_0$. Using (4.3) we see that $w_h \in \mathcal{D}(M_h^{-1})$. From assumption 3 we deduce the inequalities

$$\|C_hw_h\|_2 \leq c\|w_h\|_1 \leq ch^p.$$  

Assumption 6 guarantees the validity of relations

$$P_{2h}CM^{-1}v_0 - C_hM_h^{-1}P_{2h}v_0 =$$

$$= C_h(M_h^{-1}P_{2h}v_0 + w_h) - (C_hM_h^{-1}P_{2h}v_0 + y_h) = C_hw_h - y_h,$$

from where

$$\|P_{2h}CM^{-1}v_0 - C_hM_h^{-1}P_{2h}v_0\| \leq c\|w_h\|_1 + \|y_h\|_2 \leq ch^p$$

and that was to be legalized. The assertion of theorem 4.2 then directly follows from theorem 3.1.

The accuracy-order of the scheme \{M_h, C_h\} for the eigenvalue problem $Mu = \lambda Cu$ can be investigated without the dominantness of the proper values $\mu_0$ and $\mu_{0,h}$. A result in this direction is contained in theorem 4.3.

An important class of problems for which theorem 4.3. can be applied is formed by positive irreducible operators $CM^{-1}$.

Suppose that the spectrum of the operator $CM^{-1}$ contains a finite number of simple poles $\mu_1, \ldots, \mu_s$ of the resolvent $R(\lambda, CM^{-1})$, where $|\mu_j| = r(CM^{-1})$ and $r(CM^{-1})$ denotes the spectral radius of $CM^{-1}$. Then to every $\mu_j$ there exists a complex number $v_j$ such that $\mu_j + v_j$ is a dominant point of the spectrum $\sigma(T_j)$, where

$$T_j = CM^{-1} + v_jI.$$

For fixed $j$ we put

$$D_j = v_jM + C, \quad L = M.$$  

Then evidently $D_jL^{-1} = T_j$ and $T_j$ has a dominant proper value $\varrho_j = \mu_j + v_j$. For solution of the problem

$$Lu_j = \sigma_jD_ju_j, \quad \sigma_j = \varrho_j^{-1}$$

theorem 4.2 can be used. It is easy to see that

$$Mu_j = \mu_j^{-1}Cu_j,$$

where $u_j$ is a proper vector of the problem (4.7).

**Theorem 4.3. Assumptions.**

1. The operators $M$ and $M_h$ which map $\mathcal{D}(M) \subset X_1$ and $\mathcal{D}(M_h) = X_{1,h}$ into $X_2$ and $X_{2,h}$ have inverses $M^{-1} \in [X_2, X_1]$ and $M_h^{-1} \in [X_{2,h}, X_{1,h}]$.

2. $\mathcal{R}(M^{-1}) \subset \mathcal{D}(C)$ and $\mathcal{R}(M_h^{-1}) \subset \mathcal{D}(C_h)$.
3. There exists a constant $c$ such that
\begin{equation}
\|Cy\|_2 \leq c\|y\|_1,
\end{equation}
for $y \in \mathcal{R}(M^{-1}_-)$ and
\begin{equation}
\|C_h y_h\|_2 \leq c\|y_h\|_1
\end{equation}
for $y_h \in \mathcal{R}(M^{-1}_h)$.

4. For every $h \in H$
\begin{equation}
P_{1h} M^{-1}_- \mathcal{X}_2 \subset M^{-1}_h \mathcal{X}_{2,h}.
\end{equation}

5. The resolvents $R(\lambda, CM^{-1})$ and $R(\lambda, C_h M^{-1}_h)$ have $s$ and $s_h = s(h)$ simple poles $\mu_1, \ldots, \mu_s$ and $\mu_1^{(h)}, \ldots, \mu_s^{(h)}$ on the circles $|\lambda| = r(CM^{-1})$ and $|\lambda| = r(C_hM^{-1}_h)$ respectively.

If $v'_j$ is a proper vector of $CM^{-1}$ corresponding to $\mu_j$, then we put $v^{(h)}_j = B^{(h)}_j hP_{2h}v'_j$, where $B^{(h)}_j$ is the unique coefficient of the main part of Laurent expansion of $R(\lambda, C_hM^{-1}_h)$ in a neighbourhood of $\mu_s^{(h)}$.

6. For every vector $u \in \mathcal{R}(M^{-1})$ there exists vector $y_h \in \mathcal{X}_{2,h}$ such that
\[ C_h P_{1h} u = P_{2h} C u + y_h \]
and
\[ \|y_h\| \leq ch^p, \]

7. The operators $C_h M^{-1}_h$, $h \in H$, are uniformly bounded: $\|C_h M^{-1}_h\| \leq c$.

8. There are forms $\xi_{h,j,k} \in \mathcal{X}_{2,h}$, $h \in H$, such that
\begin{enumerate}
\item $\|\xi_{h,j,k}\| \leq c$,
\item $\xi_{h,j,k}(C_h M^{-1}_h x_h) = \mu^{(h)}_j \xi_{h,j,k}(x_h)$, $x_h \in \mathcal{X}_{2,h}$,
\item $|\xi_{h,j,k}(P_{2h} v'_j)| \geq c > 0$, $j = 1, \ldots, s$; $k = 1, \ldots, s_h$.
\end{enumerate}

9. The scheme $\{M_h, I_h\}$ has the accuracy-order $p$ in $\mathcal{X}_2$ with respect to the problem $Mu = v$.

10. For every $j$, $j = 1, \ldots, s$, there is a complex $\nu_j$ such that the operators $T_j = CM^{-1}_ - \nu_j I$, $T_{h,j} = C_h M^{-1}_h + \nu_j I_h$ have dominant proper values $\mu_j + \nu_j$ and $\mu^{(h)}_j + \nu_j$ for suitable $k$.

Assertion. The scheme $\{M_h, C_h; \lambda_1, \ldots, \lambda_s\}$, where $\lambda_j = \mu^{-1}_j$, $j = 1, \ldots, s$, has the accuracy-order $p$ for the characteristic values $\lambda_1, \ldots, \lambda_s$ of the problem $Mu = \lambda Cu$.

Proof. Let us choose $j$ fixed and let us investigate operators $T_j$, $T_{h,j}$ together with problems
\begin{equation}
Lu = \sigma D_j u, \quad L_h u_h = \sigma_h D_{h,j} u_h,
\end{equation}

102
where $L$ and $D_j$ are defined by (4.6) and

\[ L_h = M_h, \quad D_{h,j} = v_j M_h + C_h. \]

We shall prove that for constructing $G_j = Q_j \pi_1 M^k + v_j$ and corresponding proper vectors, theorem 4.2 can be applied. From this the validity of theorem 4.3 will follow.

Similarly as in proofs of theorems 4.1 and 4.2 it is sufficient to legalize only the fulfillment of the assumption (c) of theorem 3.1 for the operator $T_j$.

Let us consider vectors $P_{2h} T_j v_j - T_{h,j} P_{2h} v_j$, where $v_j = \mu_j^{-1} C M^{-1} v_j$. Assumption 10 guarantees the dominantness of $\mu_j + v_j$ and $\mu^{(h)}_k + v_j$. Assumption 9 gives a constant $c$ such that $\|w_h\| \leq c h^p$, where $w_h = P_{1h} M^{-1} v_j - M^{-1} P_{2h} v_j$. According to (4.11) $w_h \in \mathcal{R}(L^{-1}_h)$. Assumption 3 implies the inequalities

\[ \|C_h w_h\|_2 \leq c \|w_h\|_1 \leq c h^p. \]

From definition of $T_j$ and $T_{h,j}$ using assumption 6 we obtain the relations

\[
P_{2h} T_j v_j - D_{h,j} L^{-1}_h P_{2h} v_j =
P_{2h} (C + v_j M) M^{-1} v_j - (C_h + v_j M_h) M^{-1}_h P_{2h} v_j =
C_h P_{1h} M^{-1} v_j - y_h - C_h M^{-1}_h P_{2h} v_j.
\]

All these considerations yield to the relations $\|P_{2h} T_j v_j - D_{h,j} L^{-1}_h P_{2h} v_j\| \leq \|C_h w_h\| + \|y_h\|$ and consequently to the required inequality $\|P_{2h} T_j v_j - T_{h,j} P_{2h} v_j\| \leq c h^p$. In other words the requirement (c) of theorem 3.1 is fulfilled. From this theorem we conclude that

\[ |\mu_j - \mu^{(h)}_k| \leq |(\mu_j + v_j) - (\mu^{(h)}_k + v_j)| \leq c h^p, \quad \|P_{2h} v_j - B^{(h)}_{1h} P_{2h} v_j\| \leq c h^p
\]

and these estimates show that the scheme $\{M_h, C_h; \mu_j^{-1}\}$ has the accuracy-order $p$ in $\mathcal{X}_2$.

References


Author's address: Praha 8 - Karlin, Sokolovská 83 (Matematicko-fyzikální fakulta KU).
Výtah

K TEORII PŘIBLIŽNÝCH METOD
V PROBLÉMECH VLASTNÍCH HODNOT

IVO MAREK, Praha

Vyšetřují se problémy vlastních hodnot pro rovnice typu $M u = \lambda C u$, $M_h u_h = \lambda^{(h)} C_h u_h$, kde $u \in X$, $M u$, $C u \in Y$; $u_h \in X_h$, $L_h u_h$, $C_h u_h \in Y_h$ při čemž $X$, $Y$ resp. $X_h$, $Y_h$ jsou Banachovy prostory a $M$, $C$ resp. $M_h$, $C_h$ lineární zobrazení zobrazující husté podmnožiny $X$ resp. $X_h$ do $Y$ resp. $Y_h$. Jmenovitě se vyšetřuje řád přesnosti „přibližných“ vlastních prvků $u_h$, $\lambda^{(h)}$ vzhledem k vlastním prvkům určeným pomocí operátorů $L$ a $C$. Jsou uvedeny postačující podmínky, jež zaručují, že řád přesnosti pro úlohu na vlastní hodnoty je roven řádu přesnosti odpovídající úlohy nehomogenní (princip dehomogenisace).

РЕЗЮМЕ

О ПРИБЛИЖЕННЫХ МЕТОДАХ В ЗАДАЧАХ О СОБСТВЕННЫХ ЗНАЧЕНИЯХ

ИВО МАРЕК (Ivo Marek), Прага

В статье рассматриваются проблемы о собственных значениях для уравнений типа $M u = \lambda C u$, где $u \in X$, $M u$, $C u \in Y$ и $X$, $Y$ — банаховы пространства и $M$, $C$ — линейные отображения плотных областей определения из $X$ в $Y$. Вместе с этой задачей рассматривается „приближенная“ проблема $M_h u_h = \lambda^{(h)} C_h u_h$, где линейные операторы $M_h$, $C_h$ отображают плотные подмножества из банахова пространства $X_h$ в банахово пространство $Y_h$. Особенно исследован порядок точности „приближенных“ собственных элементов $u_h$, $\lambda^{(h)}$ задачи $M_h u_h = \lambda^{(h)} C_h u_h$ относительно точных собственных элементов уравнения $M u = \lambda C u$. Приводятся достаточные условия обеспечивающие равенство порядков точности задачи о собственных значениях и соответствующей неоднородной задачи. Эти исследования являются базой так называемого принципа дегомогенизации.