Časopis pro pěstování matematiky

Miloslav Jiřina

On Feller's branching diffusion processes

Časopis pro pěstování matematiky, Vol. 94 (1969), No. 1, 84--90

Persistent URL: `http://dml.cz/dmlcz/117652`

ON FELLER'S BRANCHING DIFFUSION PROCESSES

MILOSLAV JIŘINA, Praha

(Received November 24, 1967)

In [1] the following limiting procedure for branching processes is described: Let in any one-type-particle branching process with finite second moments the time parameter and the states of the process be transformed for each n in such a way that one new time unit corresponds to n time units of the original process and one unit in the new state space corresponds to n particles of the original process. Then, according to [1], the sequence of the transformed processes converges with $n \to \infty$ to a branching diffusion process whose basic transition probabilities $Q(t, A)$ are absolutely continuous on $(0, \infty)$ with the densities

$$q(t, a) = \exp\left[\frac{\alpha(e^{\alpha t} + a)}{\beta(e^{\alpha t} - 1)} \sum_{\nu=0}^{\infty} \frac{1}{\nu!\,(\nu + 1)!} \left(\frac{\alpha^2 e^{\alpha t}}{\beta^2(e^{\alpha t} - 1)^2}\right)^{\nu+1} a^{\nu}\right] \tag{1}$$

and with

$$Q(t, \{0\}) = \exp\left[\frac{\alpha e^{\alpha t}}{\beta(e^{\alpha t} - 1)}\right],$$

where $-\infty < \alpha < \infty$ and $0 < \beta < \infty$; $\alpha/(e^{\alpha t} - 1)$ is to be replaced by $1/t$ if $\alpha = 0$.

The density $q(t, a)$ satisfies the Kolmogorov equation

$$\frac{\partial}{\partial t} q(t, a) = \beta \frac{\partial^2}{\partial a^2}(aq(t, a)) - \alpha \frac{\partial}{\partial a}(aq(t, a)) \tag{2}$$

and the corresponding characteristic function $f(t, y)$ satisfies

$$\frac{\partial}{\partial t} f(t, y) = (\alpha y + i\beta y^2) \frac{\partial}{\partial y} f(t, y)\,. \tag{3}$$

(See [1], (5.1), (5.2), (5.6) and (12.9).)

The proof of the assertion stated above, as presented in [1], is not complete. It shows only that the limit $f(t, y)$ of the characteristic functions of the transformed processes satisfies (3), provided that the limit exists in such a way that not only the characteristic functions of the transformed processes converge to $f(t, y)$, but also their relative differences converge to the derivatives of $f(t, y)$. The existence of these limits is not proved in [1]. It is also not clear in [1] how the first and second moments of the original process are to be changed before the n-th transformation is applied. In the preliminary discussion in [1] (page 236) the moments are assumed to have the form $1 + \alpha/n$, β/n; in the proof of (12.9) ([1] page 245), the second moment is constant. To obtain the same results for the first and second moment (see [1], (5.11), (5.12)), it is necessary to leave the particles at the beginning of the process unchanged for the first alternative and to transform them in the same way as the states for the second alternative. The asymptotic behaviour of the third moments indicates that only the second alternative can lead to reasonable results.

Asymptotic properties of branching processes with transformed time and states were studied in several papers by Lamperti (see [2], e.g.). It seems, however, that the original assertion of [1] has not yet been proved completely. It is the purpose of this paper to provide a correct proof for this assertion.

Since all probability distributions in this paper are concentrated on the non-negative part of the real line, it is somewhat more suitable to use real Laplace transforms or their logarithms instead of characteristic functions. We shall not be interested in differential equations, but we shall prove directly that the logarithm of the Laplace transform of the transition probabilities for the n-th transformed process converge (with $n \to \infty$) to $\psi(t, x)$ defined by (7), which is the desired result (see remark following the Theorem).

Let $\mathscr{P}_n$ be a sequence of homogeneous Markov branching processes with one type of particles, with discrete time parameter $t \in T = \{0, 1, 2, \ldots\}$ and with the state space $S = \{0, 1, 2, \ldots\}$. The probability of the transition from the state a to the state b after t time units in the n-th process $\mathscr{P}_n$ will be denoted by $P_n(t, a, b)$. The probability distribution induced by $P_n(t, a, b)$ on S will be denoted by $P_n(t, a, .)$. We shall suppose that each $\mathscr{P}_n$ has finite third moments and we shall denote by M_n and D_n the mean value and dispersion of $P_n(1, 1, .)$. To each $\mathscr{P}_n$ we shall assign a new Markov process $\mathscr{Q}_n$ with discrete time parameter $t \in T_n = \{0, 1/n, 2/n, \ldots\}$ and with the state space $S_n = \{0, 1/n, 2/n, \ldots\}$. The transition probabilities $Q_n(t, a, b)$ of $\mathscr{Q}_n$ will be defined by

$$(4) \qquad Q_n(t, a, b) = P_n(tn, an, bn)\,, \quad t \in T_n\,, \quad a, b \in S_n\,.$$

It is easily seen that $\mathscr{Q}_n$ is really a Markov process and it is also a branching process in the sense that for each $t \in T_n$ and $a \in S_n$

$$Q_n(t, a, .) = Q_n(t, 1/n, .)^{*an}\,,$$

where $Q_n(t, a, .)$ denotes the probability distribution induced on S_n by $Q_n(t, a, b)$ and the symbol $*k$ indicates that the operation of convolution is to be applied k-times.

Let $\Phi_n(t, a, x)$ be the Laplace transform of the probability distribution $P_n(t, a, .)$

and $\varphi_n(t, a, x)$ its logarithm, i.e. $\Phi_n(t, a, x) = \sum_{b=0}^{\infty} e^{xb} P_n(t, a, b)$ and $\varphi_n(t, a, x) = \log \Phi_n(t, a, x)$ for all $t \in T$, $a \in S$ and $x \leqq 0$. Similarly

$$\psi_n(t, a, x) = \log \sum_{b \in S_n} e^{xb} Q_n(t, a, b)$$

for all $t \in T_n$, $a \in S_n$ and $x \leqq 0$.

By (4)

$$(5) \qquad \psi_n(t, a, x) = \varphi_n\left(tn, an, \frac{x}{n}\right).$$

From the basic identity for branching processes

$$(6) \qquad \varphi_n(t, a, x) = a\, \varphi_n(t, 1, x)$$

it follows by (5) that a similar relation holds for $\mathcal{Q}_n$:

$$\psi_n(t, a, x) = a\, \psi_n(t, 1, x).$$

It is therefore sufficient to study the function $\psi_n(t, 1, x)$. For simplicity reasons we shall write $\psi_n(t, x)$ instead of $\psi_n(t, 1, x)$ and similarly $\Phi_n(t, x)$ and $\varphi_n(t, x)$ instead of $\Phi_n(t, 1, x)$ and $\varphi_n(t, 1, x)$. We shall also ommit the value $t = 1$; hence $\Phi_n(x) = \Phi_n(1, 1, x)$, $\varphi_n(x) = \varphi_n(1, 1, x)$.

For each real $t \geqq 0$ let $[t]_n$ denote the largest $\tau \in T_n$ less or equal to t, i.e. $[t]_n = [tn]/n$, where $[y] = [y]_1$ denotes the integral part of y.

Theorem. *Let us suppose that the limits*

$$(7) \qquad \lim_{n \to \infty} n(M_n - 1) = \alpha\,, \quad \lim_{n \to \infty} \tfrac{1}{2} D_n = \beta$$

exist with $-\infty < \alpha < \infty$, $0 < \beta < \infty$ *and that the third moments* $\Phi_n'''(0)$ *are bounded with respect to n. Then for each fixed* $t \geqq 0$ *and* $x_0 \leqq 0$

$$(8) \qquad \psi_n([t]_n, x) \xrightarrow[n \to \infty]{} \psi(t, x) =$$

$$= \begin{cases} x e^{\alpha t} \left[1 + \dfrac{\beta}{\alpha}(1 - e^{\alpha t})\, x\right]^{-1} & \text{if } \alpha \neq 0 \\ x[1 - \beta t x]^{-1} & \text{if } \alpha = 0 \end{cases}$$

uniformly with respect to $x \in \langle x_0, 0 \rangle$.

Remark. It is easily seen by straightforward calculation that $\psi(t, x)$ is the logarithm of the Laplace transform of the probability distribution $Q(t, a)$ defined by (1). It follows then from (8) that $Q_n([t]_n, .)$ converges (weakly) to $Q(t, .)$.

Proof of the Theorem. It follows from (7) that $M_n > 0$ for all sufficiently large n. Hence, we may assume without loss of generality that $M_n > 0$ for all n. Let us write $\alpha_n = n(M_n - 1)$. Then $\alpha_n \underset{n}{\to} \alpha$ according to (7). The proof of the Theorem will consist of several lemmas. In all of them the real number $t \geqq 0$ is fixed.

(I) $$M_n^{[tn]} = \left(1 + \frac{\alpha_n}{n}\right)^{[tn]} \underset{n}{\to} e^{\alpha t}.$$

(II) If $\alpha_n \neq 0$, $\alpha_n \underset{n}{\to} 0$, then

$$\frac{M_n^{[tn]} - 1}{n(M_n - 1)} = \frac{\left(1 + \frac{\alpha_n}{n}\right)^{[tn]} - 1}{\alpha_n} \underset{n}{\to} t.$$

(III) There exists n_0 such that

$$M_n^s \leqq e^{2|\alpha| t}$$

for all $n \geqq n_0$ and all $s = 0, 1, \dots [tn]$.

Proof of (III). Since $\alpha_n \underset{n}{\to} \alpha$, there exists n_0 such that $|\alpha_n| \leqq 2|\alpha|$ for $n \geqq n_0$. Then for $n \geqq n_0$ and $s \leqq [tn]$

$$M_n^s = \left(1 + \frac{\alpha_n}{n}\right)^s \leqq \left(1 + \frac{2|\alpha|}{n}\right)^s \leqq \left(1 + \frac{2|\alpha|}{n}\right)^{[tn]} \leqq \left(1 + \frac{2|\alpha|}{n}\right)^{tn} \leqq e^{2|\alpha| t}.$$

(IV) There exists n_0 such that

$$0 \geqq \varphi_n(s, x) \geqq e^{2|\alpha| t} x$$

for all $n \geqq n_0$, all $s = 0, 1, \dots [tn]$ and all $x \leqq 0$.

Proof of (IV). It is well known for branching processes that

(9) $$\varphi_n'(s, 0) = \Phi_n'(s, 0) = M_n^s \quad \text{for all} \quad s \in T.$$

Further

(10) $$\varphi_n''(s, x) = \frac{\Phi_n''(s, x)\, \Phi_n(s, x) - [\Phi_n'(s, x)]^2}{\Phi_n^2(s, x)} =$$

$$= [(\sum_{b=0}^{\infty} b^2 e^{xb} P_n(s, b)) (\sum_{b=0}^{\infty} e^{xb} P_n(s, b)) - (\sum_{b=0}^{\infty} b e^{xb} P_n(s, b))^2]\, \Phi_n^{-2}(s, x).$$

Hence, by Schwartz inequality,

(11) $$\varphi_n''(s, x) \geqq 0 \quad \text{for all} \quad s \in T \quad \text{and} \quad x \leqq 0.$$

Using Taylor formula, (9) and (11) we have

(12) $$0 \geqq \varphi_n(s, x) = M_n^s x + \tfrac{1}{2} \varphi_n''(s, \xi)\, x^2 \geqq M_n^s x$$

and (IV) follows from (III) for the same n_0.

(V) There exists $\delta_0 < 0$ and $K_0 > 0$ such that

$$\varphi_n''(x) < K_0\,, \quad |\varphi_n'''(x)| < K_0$$

for all $\delta_0 < x \leqq 0$ and all n.

Proof of (V). According to the assumptions of the theorem, there exists $K_1 > 0$ such that $0 \leqq \Phi_n'(x) \leqq \Phi_n'(0) = M_n < K_1$, $0 \leqq \Phi_n''(x) \leqq \Phi_n''(0) = D_n + M_n^2 < K_1$, $0 \leqq \Phi_n'''(x) \leqq \Phi_n'''(0) < K_1$ for all $x \leqq 0$ and all n. Using (12) with $s = 1$ we see that $|\varphi_n(x)| < K_1|x|$ for all n and, consequently there exists $\delta_0 < 0$ such that $\Phi_n(x) > > \frac{1}{2}$ for all $\delta_0 < x \leqq 0$ and all n. The first inequality of (V) follows now from (10). The proof of the second inequality is similar.

(VI)
$$\lim_{\substack{x \to 0 \\ n \to \infty}} \varphi_n''(x) = 2\beta\,.$$

Proof of (VI). By (V),

$$|\varphi_n''(x) - 2\beta| \leqq |\varphi_n''(x) - \varphi_n''(0)| + |\varphi_n''(0) - 2\beta| =$$
$$= |\varphi_n'''(\eta_n)\,x| + |D_n - 2\beta| \leqq K_0|x| + |D_n - 2\beta|$$

for all $\delta < x \leqq 0$ and all n; the assertion now follows from the assumption $D_n \underset{n}{\to} 2\beta$.

(VII)
$$\varphi_n(s, x) < 0 \quad \text{for all } n, \quad s \in T \quad \text{and} \quad x < 0\,.$$

Proof of (VII). Let us suppose to the contrary that $\varphi_n(s, x_1) = 0$ for some n, s and $x_1 < 0$. Then $\varphi_n(s, x) = 0$ for all $x_1 \leqq x \leqq 0$, because $\varphi_n(s, x)$ is non-decreasing and $\varphi_n(s, 0) = 0$. Hence, by (9), $M_n^s = \varphi_n'(s, 0) = 0$, which contradicts the assumption $M_n > 0$.

(VIII) To each $\varepsilon > 0$ there exist n_1 and $\delta_1 < 0$ such that

$$\left|1 - \frac{\varphi_n(s-1, x)}{\varphi_n(s, x)}\right| < \varepsilon$$

for all $n \geqq n_1$, all $s = 1, \ldots, [tn]$ and all $\delta_1 \leqq x < 0$.

Proof of (VIII). It is well known that

(13)
$$\varphi_n(s, x) = \varphi_n(\varphi_n(s-1, x))$$

holds for branching processes (see [3], Chap. III, § 7, e.g.). Using (13) and the Taylor formula we have

(14)
$$\varphi_n(s, x) = M_n\,\varphi_n(s-1, x) + \tfrac{1}{2}\varphi_n''(\zeta_n(s, x))\,\varphi_n^2(s-1, x)$$

where, by (VII),

(15)
$$\varphi_n(s-1, x) < \zeta_n(s, x) < 0 \quad \text{for} \quad x < 0\,.$$

Hence we may write for $x < 0$

$$1 - \frac{\varphi_n(s, x)}{\varphi_n(s-1, x)} = -\left(\frac{\alpha_n}{n} + \frac{1}{2}\,\varphi_n''(\zeta_n(s, x))\,\varphi_n(s-1, x)\right).$$

Let us choose $\delta_0 < 0$ according to (V) and put $\delta_0' = e^{-2|\alpha|t}\delta_0$. Then, for all $n \geqq n_0$, all $s = 1, 2, \ldots, [tn]$ and all $\delta_0' \leqq x < 0$,

$$\delta_0 \leqq \varphi_n(s-1, x) < \zeta_n(s, x) < 0$$

and

$$\left|1 - \frac{\varphi_n(s, x)}{\varphi_n(s-1, x)}\right| \leqq \frac{\alpha_n}{n} + \frac{1}{2}K_0 e^{2|x|t}|x|$$

according to (IV) and (V).

The assertion of (VIII) follows now from the fact that the right-hand side of the last inequality is arbitrarily small for sufficiently large n and sufficiently small $|x|$.

We shall now finish the proof of the Theorem. Let us write for $s \leqq s_0$

$$F_n(s, s_0, x) \leqq M_n^{s_0-s+1}\,\varphi_n(s-1, x) - M_n^{s_0-s}\,\varphi_n(s, x)\,.$$

The assertion of the Theorem is trivial for $x = 0$. We shall assume $x_0 \leqq x < 0$ and then, by (VII), $\varphi_n(s-1, x) < 0$. Hence, we may write then

$$(16) \qquad \frac{1}{M_n^{s_0-s}\,\varphi_n(s, x)} = \frac{1}{M_n^{s_0-s+1}\,\varphi(s-1, x)} + G_n(s, s_0, x)$$

where

$$(17) \qquad G_n(s, s_0, x) = F_n(s, s_0, x)/M_n^{2s_0-2s+1}\,\varphi_n(s-1, x)\,\varphi_n(s, x) = \\ = -\tfrac{1}{2}\varphi_n''(\zeta_n(s, x))\,\frac{\varphi_n(s-1, x)}{\varphi_n(s, x)}\,M_n^{-(s_0-s+1)}$$

according to (14). Summing in (16) for $s = 1, \ldots, s_0$ we obtain

$$\frac{1}{\varphi_n(s_0, x)} = \frac{1}{M_n^{s_0}x} + \sum_{s=1}^{s_0} G_n(s, s_0, x)$$

and, finally, by (5), (6) and (17),

$$(18) \qquad \frac{1}{\psi_n([t]_n, x)} = \frac{1}{n\,\varphi_n\left([tn], \frac{x}{n}\right)} =$$

$$= \frac{1}{M_n^{[tn]}}\left[\frac{1}{x} - \frac{1}{2n}\sum_{s=0}^{[tn]}\varphi_n''\left(\zeta_n\left(s, \frac{x}{n}\right)\right)\frac{\varphi_n\left(s-1, \frac{x}{n}\right)}{\varphi_n\left(s, \frac{x}{n}\right)}M_n^{s-1}\right].$$

Let $\varepsilon > 0$. By (III), (IV), (VI), (VIII) and (15), there exists n_2 such that

$$\left| \frac{1}{2} \varphi_n''\left(\zeta_n\left(s, \frac{x}{n}\right)\right) \frac{\varphi_n\left(s-1, \frac{x}{n}\right)}{\varphi_n\left(s, \frac{x}{n}\right)} M_n^{s-1} - \beta M_n^{s-1} \right| < \varepsilon$$

for all $n > n_2$, all $s = 0, 1, \ldots, [tn]$ and all $x_0 \leqq x < 0$. Then also

$$\left| \frac{1}{2n} \sum_{s=0}^{[tn]} \varphi_n''\left(\zeta\left(s, \frac{x}{n}\right)\right) \frac{\varphi_n\left(s-1, \frac{x}{n}\right)}{\varphi_n\left(s, \frac{x}{n}\right)} M_n^{s-1} - \beta \frac{M_n^{[tn]} - 1}{n(M_n - 1)} \right| < \varepsilon$$

for all $n > n_2$ and all $x_0 \leqq x < 0$; the second ratio is to be replaced by $[tn]/n$ if $M_n = 1$ for some n. Hence, by (I), (II) and (18)

$$\frac{1}{\psi_n([t]_n, x)} \xrightarrow[n\to\infty]{} \begin{cases} e^{-\alpha t}\left[\dfrac{1}{x} - \dfrac{\beta}{\alpha}(e^{\alpha t} - 1)\right] & \text{if } \alpha \neq 0 \\ \dfrac{1}{x} - \beta t & \text{if } \alpha = 0 \end{cases}$$

uniformly in $\langle x_0, 0)$, which proves (8).

References

[1] *W. Feller:* Diffusion processes in genetics. Proc. of the 2nd Berkeley symp. on math. stat. and prob., 1951, 227–246.
[2] *J. W. Lamperti:* The limit of a sequence of branching processes. Zeitschrift für Wahrscheinlichkeitst. und verw. Geb. *7*, 1967, 271–288.
[3] *T.E. Harris:* The theory of branching processes. Springer-Verlag 1963.

Author's address: Praha 1, Žitná 25 (Matematický ústav ČSAV).