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ON THE NUMBER OF SPANNING TREES OF FINITE GRAPHS

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Under a spanning tree of a given finite graph \( G \) we understand the maximal tree subgraph of \( G \). The notation \( G = [U, H] \) means that the graph \( G \) has the vertex set \( U \) and the edge set \( H \).

Let \( G \) be a graph of \( n \) vertices, and \( k(G) \) the number of all its spanning trees\(^1\). In the paper \([5]\) we defined the set \( A_n \) as follows: It is the set of such positive integers \( q \) that there exists a connected graph \( G \) of \( n \) vertices and \( q \) spanning trees. We saw that \( |A_1| = |A_2| = 1, |A_3| = 2, |A_4| = 5, |A_5| = 16 \), where \( |M| \) denotes the number of elements of a (finite) set \( M \). In this contribution we shall investigate in the first place the behaviour of the number \( |A_n| \) for large values of \( n \).

Theorem 1.

\[
\lim_{n \to \infty} \frac{|A_n|}{n} = +\infty.
\]

Proof. Let us put \( U = \{u_1, u_2, \ldots, u_n\} \) and consider the set \( A_n = \{x_1, x_2, \ldots, x_r\} \) where \( x_1 < x_2 < \ldots < x_r \). Let \( G_r = [U, H_r] \) denote a complete graph of \( n \) vertices; thus \( k(G_r) = x_r \).

Firstly we shall now deduce an inequality between the numbers \( |A_{n+1}| \) and \( |A_n| \), \( n \geq 2 \). We know that \( A_n \subset A_{n+1} \). Let us choose a vertex \( v \) non \( \in U \) and a positive integer \( s \in \langle 2, n \rangle \) and define a graph \( G^{(s)} = [U^*, H^{(s)}] \) such that \( U^* = U \cup \{v\} \), \( H^{(s)} = H_r \cup \{u_1v, u_2v, \ldots, u_nv\} \). Using a well-known determinant method (see, e.g. \([1]\)) we may compute that \( k(G^{(s)}) = sn^{s-2}(n+1)^{s-1} \). Then

\[
x_r = n^{r-2} < k(G^{(2)}) < k(G^{(3)}) < \ldots < k(G^{(n)}) = (n + 1)^{n-1},
\]

where \( k(G^{(s)}) \in A_{n+1} \) for every \( s = 2, 3, \ldots, n \). Also \( |A_{n+1}| \geq |A_n| + n - 1 \). By

\(^1\) J. W. Moon \([3]\) denotes the number \( k(G) \) as the complexity of \( G \).
In the following theorem we shall show that LUCAS numbers — well known in the theory of numbers — play a definite role in the graph theory, see e.g. [4]. First, let us recall this notion. Let be given an equation \( z^2 - Lz + M = 0 \), \( L > 0 \) and \( M \) being integers and \( \alpha, \beta, \alpha \neq \beta \) the roots of this equation. Then the Lucas number is defined by

\[
L_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad (m = 1, 2, 3, \ldots)
\]

Now we shall consider a graph \( \mathcal{X}_m \) of \( 2m \) vertices which is illustrated in Fig. 1. We shall meet this graph in our further considerations but now we shall deduce the following theorem.

**Theorem 2.**

\[
k(\mathcal{X}_m) = \frac{1}{2} \left( (2 + \sqrt{3})^m - (2 - \sqrt{3})^m \right).
\]

**Proof.** The number of spanning trees of the graph \( \mathcal{X}_m \) may be expressed easily by a determinant but it is not evident at first sight that its value is the Lucas number of our theorem. For this reason we shall use another method.

Let us put \( a_m = k(\mathcal{X}_m) \) and denote by \( b_m \) the number of subgraphs \( \mathcal{P} \) of the graph \( \mathcal{X}_m \) wit the following properties: \( \mathcal{P} \) contains all vertices of the graph \( \mathcal{X}_m \) and consists of two components which are trees, one of them containing \( u_m \) and the other \( v_m \). We see that \( a_1 = 1, b_1 = 1 \) and the following recurrent equations hold:

\[
a_{m+1} = 3a_m + b_m, \quad b_{m+1} = 2a_m + b_m.
\]

Now we put

\[
f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i, \quad g(\xi) = \sum_{i=1}^{\infty} b_i \xi^i.
\]

2) Theorem 1 shows that the number of elements of the set \( A_n \) increases "very rapidly", but it is not clear, for instance, what is the behaviour of the fraction \( |A_n|/n^2 \) when \( n \) tends to infinity.
Considering (3) we have
\[(1 - 3\xi)f(\xi) - \xi g(\xi) = \xi, \quad 2\xi f(\xi) + (\xi - 1) g(\xi) = -\xi\]
and hence
\[f(\xi) = \frac{\xi}{\xi^2 - 4\xi + 1}, \quad g(\xi) = \frac{\xi - \xi^2}{\xi^2 - 4\xi + 1}.
\]
If we put \(\alpha = 2 + \sqrt{3}, \beta = 2 - \sqrt{3}\) then
\[f(\xi) = \frac{1}{\alpha - \beta} \left( \frac{\beta^{-1}\xi}{1 - \beta^{-1}\xi} - \frac{\alpha^{-1}\xi}{1 - \alpha^{-1}\xi} \right) = \sum_{i=1}^{\infty} \frac{\alpha^i - \beta^i}{\alpha - \beta} \xi^i.
\]
From this equation the formula (2) follows immediately. The theorem is proved.

If two positive integers \(n, t\) are given, we may define the set \(\mathcal{B}_n^{(t)}\) as a set of all positive integers \(q\) such that there exists a connected regular graph of degree \(t\) of \(n\) vertices and \(q\) spanning trees. The cases \(t = 1\) and \(t = 2\) are trivial and only if \(t \geq 3\) the problem is of some interest. Since a regular graph of degree 3 always contains an even number of vertices, \(\mathcal{B}_n^{(3)} = \emptyset\) for every odd \(n\). We may also see that \(\mathcal{B}_2^{(3)} = \emptyset\) and that for every even \(n > 2\), \(\emptyset \neq \mathcal{B}_n^{(3)} \subset A_n\).

We may conjecture by intuition that the number of elements of a set \(\mathcal{B}_n^{(3)}\) tends to infinity when \(a\) runs through the set of all positive integers. We shall prove it in the following theorem.

**Theorem 3.**

\[\lim_{a \to \infty} |\mathcal{B}_n^{(3)}| = +\infty.\]

**Proof.** Let us choose a positive integer \(a > 13\) and distinguish the following two cases:

\[Z_{a-2j-10}\]

\[j\text{-times}\]

\[\text{Fig. 2.}\]

\(^3\text{The author inserted Theorems 1 and 2 into his communication presented at the International Symposium in Manebach, May 1967.}\]
a) When \( a \) is odd, let us choose such a non-negative integer \( j \) that \( j \leq \frac{1}{2}(a - 11) \) and define a graph \( G(a,j) \) as follows: Let us construct a graph \( G_m \) of Theorem 2, for \( m = a - 2j - 10 \) and complete it in the way shown in Fig. 2. We see that for such a regular graph \( G(a,j) \) of degree 3 the equation

\[
k(G(a,j)) = 24^j \cdot k(G_{a-2j-10})
\]

holds. If for some \( j' < j \) there were \( k(G(a,j')) = k(G(a,j')) \), we should obtain after a slight modification

\[
k(G_{a-2j'-10}) = 8^{j' - j} \cdot k(G_{a-2j'-10})
\]

However that is impossible because Lucas numbers \( k(G_m) \) are odd for odd \( m \)'s. Thus we have obtained \( \frac{1}{2}(a - 9) \) different positive integers \( k(G(a,0)), k(G(a,1)), k(G(a,2)), \ldots \), therefore we may conclude that

\[
|B^{(3)}_{2a}| \geq \frac{1}{2}(a - 9).
\]

Fig. 3.

b) When \( a \) is even, let us choose again such a non-negative integer \( j \) that \( j \leq \frac{1}{2}(a - 14) \) and define a regular graph \( G(a,j) \) of degree 3 as follows: Let us construct \( G_m \) for \( m = a - 2j - 13 \) and complete it in the way shown in Fig. 3. We see that evidently

\[
k(G(a,j)) = 24^j \cdot k(G_{a-2j-13})
\]

holds for such graphs \( G(a,j) \). Thus we have constructed \( \frac{1}{2}(a - 12) \) different positive integers

\[
k(G(a,0)), k(G(a,1)), k(G(a,2)), \ldots ,
\]

hence

\[
|B^{(3)}_{2a}| \geq \frac{1}{2}(a - 12).
\]

Summing up (4) and (5) we see that (5) holds for every positive integer \( a > 13 \). From this the assertion of Theorem 3 follows and the proof is completed.\(^4\)

\(^4\) A more general theorem will be deduced in the next paper of the author (footnote added in the galley-proof on March 26, 1969).
We conclude by a remark. In the paper [5] we have presented an open problem of determining the maximal or minimal number of spanning trees of a connected regular graph of degree 3 of a given number of vertices. If we denote the maximum element of the set $B_{2a}^{(3)}$ by $y_{\text{max}}$ and if $G_{\text{max}}$ is the corresponding graph of $2a$ vertices and $y_{\text{max}}$ spanning trees, then we may express $y_{\text{max}}$ in the well-known way by means of a symmetrical determinant of degree $2a - 1$; all elements of its main diagonal are the numbers 3, outside the diagonal, $-1$ occurs three times in every row (with the exception of three rows where $-1$ occurs twice) and all other elements are zeros. By the well-known Hadamard estimation of this determinant (see, e.g., [2], p. 133) we have $y_{\text{max}} \leq 12^{a-2} \cdot 11^{3/2}$. Of course, the Hadamard estimation may be applied to any regular graph of degree $t$ (for $t \geq 3$).

References


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