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ON THE NUMBER OF SPANNING TREES OF FINITE GRAPHS

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Under a *spanning tree* of a given finite graph $\mathcal{G}$ we understand the maximal tree subgraph of $\mathcal{G}$. The notation $\mathcal{G} = [U, H]$ means that the graph $\mathcal{G}$ has the vertex set U and the edge set H.

Let $\mathcal{G}$ be a graph of n vertices, and $k(\mathcal{G})$ the number of all its spanning trees[1]. In the paper [5] we defined the set A_n as follows: It is the set of such positive integers q that there exists a connected graph $\mathcal{G}$ of n vertices and q spanning trees. We saw that $|A_1| = |A_2| = 1$, $|A_3| = 2$, $|A_4| = 5$, $|A_5| = 16$, where $|M|$ denotes the number of elements of a (finite) set M. In this contribution we shall investigate in the first place the behaviour of the number $|A_n|$ for large values of n.

Theorem 1.

$$\lim_{n \to \infty} \frac{|A_n|}{n} = +\infty . \tag{1}$$

Proof. Let us put $U = \{u_1, u_2, \ldots, u_n\}$ and consider the set $A_n = \{x_1, x_2, \ldots, x_r\}$ where $x_1 < x_2 < \ldots < x_r$. Let $\mathcal{G}_r = [U, H_r]$ denote a complete graph of n vertices; thus $k(\mathcal{G}_r) = x_r$.

Firstly we shall now deduce an inequality between the numbers $|A_{n+1}|$ and $|A_n|$, $n \geqq 2$. We know that $A_n \subset A_{n+1}$. Let us choose a vertex v non $\in U$ and a positive integer $s \in \langle 2, n \rangle$ and define a graph $\mathcal{G}^{(s)} = [U^*, H^{(s)}]$ such that $U^* = U \cup \{v\}$, $H^{(s)} = H_r \cup \{u_1 v, u_2 v, \ldots, u_s v\}$. Using a well-known determinant method (see, e.g. [1]) we may compute that $k(\mathcal{G}^{(s)}) = s n^{n-s-1}(n + 1)^{s-1}$. Then

$$x_r = n^{n-2} < k(\mathcal{G}^{(2)}) < k(\mathcal{G}^{(3)}) < \ldots < k(\mathcal{G}^{(n)}) = (n + 1)^{n-1} ,$$

where $k(\mathcal{G}^{(s)}) \in A_{n+1}$ for every $s = 2, 3, \ldots, n$. Also $|A_{n+1}| \geqq |A_n| + n - 1$. By

[1]) J. W. MOON [3] denotes the number $k(\mathcal{G})$ as the *complexity* of $\mathcal{G}$.

a well-known arrangement we obtain $|A_n| \geqq \frac{1}{2}(n^2 - 3n + 4)$ and from that (1) follows at once. Thus Theorem 1 is proved[2]).

In the following theorem we shall show that LUCAS numbers – well known in the theory of numbers – play a definite role in the graph theory, see e.g. [4]. First, let us recall this notion. Let be given an equation $z^2 - Lz + M = 0$, $L > 0$ and M being integers and $\alpha, \beta, \alpha \neq \beta$ the roots of this equation. Then the *Lucas number* is defined by

$$L_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad (m = 1, 2, 3, \ldots).$$

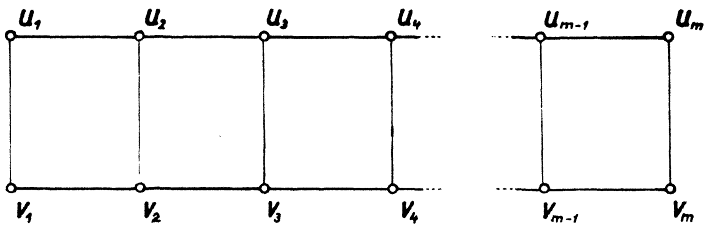

Fig. 1

Now we shall consider a graph $\mathscr{L}_m$ of $2m$ vertices which is illustrated in Fig. 1. We shall meet this graph in our further considerations but now we shall deduce the following theorem.

Theorem 2.

(2) $$k(\mathscr{L}_m) = \frac{1}{2\sqrt{3}}((2 + \sqrt{3})^m - (2 - \sqrt{3})^m).$$

Proof. The number of spanning trees of the graph $\mathscr{L}_m$ may be expressed easily by a determinant but it is not evident at first sight that its value is the Lucas number of our theorem. For this reason we shall use another method.

Let us put $a_m = k(\mathscr{L}_m)$ and denote by b_m the number of subgraphs $\mathscr{P}$ of the graph $\mathscr{L}_m$ wit the following properties: $\mathscr{P}$ contains all vertices of the graph $\mathscr{L}_m$ and consists of two components which are trees, one of them containing u_m and the other v_m. We see that $a_1 = 1$, $b_1 = 1$ and the following recurrent equations hold:

(3) $$a_{m+1} = 3a_m + b_m, \quad b_{m+1} = 2a_m + b_m.$$

Now we put

$$f(\xi) = \sum_{i=1}^{\infty} a_i \xi^i, \quad g(\xi) = \sum_{i=1}^{\infty} b_i \xi^i.$$

[2]) Theorem 1 shows that the number of elements of the set A_n increases "very rapidly", but it is not clear, for instance, what is the behaviour of the fraction $|A_n|/n^2$ when n tends to infinity.

Considering (3) we have

$$(1 - 3\xi) f(\xi) - \xi\, g(\xi) = \xi\,, \quad 2\xi f(\xi) + (\xi - 1)\, g(\xi) = -\xi$$

and hence

$$f(\xi) = \frac{\xi}{\xi^2 - 4\xi + 1}\,, \quad g(\xi) = \frac{\xi - \xi^2}{\xi^2 - 4\xi + 1}\,.$$

If we put $\alpha = 2 + \sqrt{3}$, $\beta = 2 - \sqrt{3}$ then

$$f(\xi) = \frac{1}{\alpha - \beta}\left(\frac{\beta^{-1}\xi}{1 - \beta^{-1}\xi} - \frac{\alpha^{-1}\xi}{1 - \alpha^{-1}\xi}\right) = \sum_{i=1}^{\infty} \frac{\alpha^i - \beta^i}{\alpha - \beta}\,\xi^i\,.$$

From this equation the formula (2) follows immediately. The theorem is proved[3]).

If two positive integers n, t are given, we may define the set $B_n^{(t)}$ as a set of all positive integers q such that there exists a connected regular graph of degree t of n vertices and q spanning trees. The cases $t = 1$ and $t = 2$ are trivial and only if $t \geqq 3$ the problem is of some interest. Since a regular graph of degree 3 always contains an even number of vertices, $B_n^{(3)} = \emptyset$ for every odd n. We may also see that $B_2^{(3)} = \emptyset$ and that for every even $n > 2$, $\emptyset \neq B_n^{(3)} \subset A_n$.

We may conjecture by intuition that the number of elements of a set $B_{2a}^{(3)}$ tends to infinity when a runs through the set of all positive integers. We shall prove it in the following theorem.

Theorem 3.

$$\lim_{a \to \infty} \left|B_{2a}^{(3)}\right| = +\infty\,.$$

Proof. Let us choose a positive integer $a > 13$ and distinguish the following two cases:

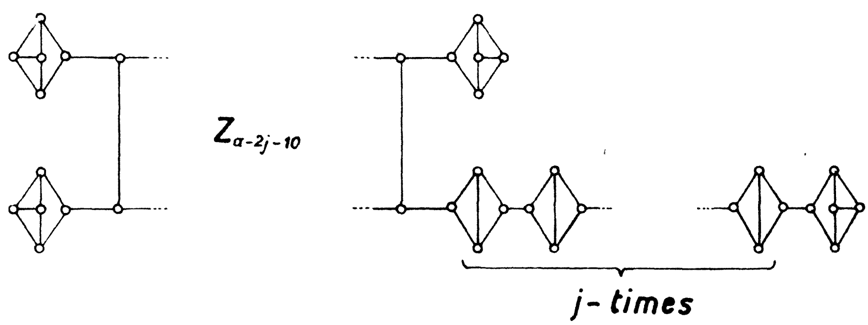

Fig. 2.

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[3]) The author inserted Theorems 1 and 2 into his communication presented at the International Symposium in Manebach, May 1967.

a) When a is odd, let us choose such a non-negative integer j that $j \leqq \frac{1}{2}(a - 11)$ and define a graph $\mathscr{G}^{(a,j)}$ as follows: Let us construct a graph $\mathscr{Z}_m$ of Theorem 2, for $m = a - 2j - 10$ and complete it in the way shown in Fig. 2. We see that for such a regular graph $\mathscr{G}^{(a,j)}$ of degree 3 the equation

$$k(\mathscr{G}^{(a,j)}) = 24^4 \,.\, 8^j \,.\, k(\mathscr{Z}_{a-2j-10})$$

holds. If for some $j' < j''$ there were $k(\mathscr{G}^{(a,j')}) = k(\mathscr{G}^{(a,j'')})$, we should obtain after a slight modification

$$k(\mathscr{Z}_{a-2j'-10}) = 8^{j''-j'} \,.\, k(\mathscr{Z}_{a-2j''-10}) \,.$$

However that is impossible because Lucas numbers $k(\mathscr{Z}_m)$ are odd for odd m's. Thus we have obtained $\frac{1}{2}(a - 9)$ different positive integers $k(\mathscr{G}^{(a,0)})$, $k(\mathscr{G}^{(a,1)})$, $k(\mathscr{G}^{(a,2)})$, ..., therefore we may conclude that

(4) $$|B_{2a}^{(3)}| \geqq \frac{1}{2}(a - 9) \,.$$

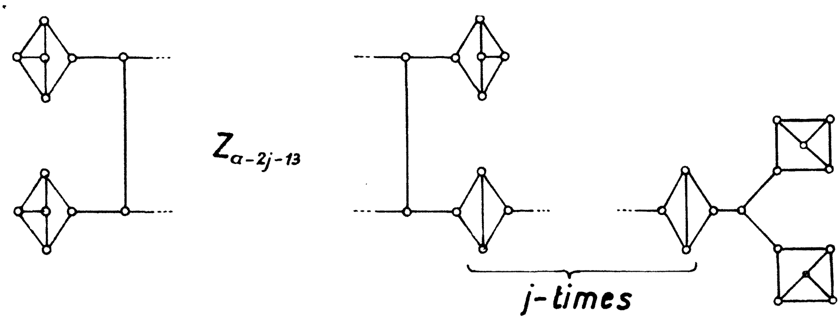

Fig. 3.

b) When a is even, let us choose again such a non-negative integer j that $j \leqq \leqq \frac{1}{2}(a - 14)$ and define a regular graph $\mathscr{G}_*^{(a,j)}$ of degree 3 as follows: Let us construct $\mathscr{Z}_m$ for $m = a - 2j - 13$ and complete it in the way shown in Fig. 3. We see that evidently

$$k(\mathscr{G}_*^{(a,j)}) = 24^5 \,.\, 8^j \,.\, k(\mathscr{Z}_{a-2j-13})$$

holds for such graphs $\mathscr{G}_*^{(a,j)}$. Thus we have constructed $\frac{1}{2}(a - 12)$ different positive integers

$$k(\mathscr{G}_*^{(a,0)}),\ k(\mathscr{G}_*^{(a,1)}),\ k(\mathscr{G}_*^{(a,2)}),\ \ldots,$$

hence

(5) $$|B_{2a}^{(3)}| \geqq \frac{1}{2}(a - 12) \,.$$

Summing up (4) and (5) we see that (5) holds for every positive integer $a > 13$. From this the assertion of Theorem 3 follows and the proof is completed.[4])

[4]) A more general theorem will be deduced in the next paper of the author (footnote added in the galley-proof on March 26, 1969).

We conclude by a remark. In the paper [5] we have presented an open problem of determining the maximal or minimal number of spanning trees of a connected regular graph of degree 3 of a given number of vertices. If we denote the maximum element of the set $B_{2a}^{(3)}$ by $y_{\max}$ and if $\mathcal{G}_{\max}$ is the corresponding graph of $2a$ vertices and $y_{\max}$ spanning trees, then we may express $y_{\max}$ in the well-known way by means of a symmetrical determinant of degree $2a - 1$; all elements of its main diagonal are the numbers 3, outside the diagonal, -1 occurs three times in every row (with the exception of three rows where -1 occurs twice) and all other elements are zeros. By the well-known HADAMARD estimation of this determinant (see, e.g., [2], p. 133) we have $y_{\max} \leqq 12^{a-2} . 11^{3/2}$. Of course, the Hadamard estimation may be applied to any regular graph of degree t (for $t \geqq 3$).

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