

Angeline Brandt; Jakub Intrator

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## THE ASSIGNMENT PROBLEM WITH THREE JOB CATEGORIES

A. BRANDT, Y. INTRATOR, Rehovot

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**1. Introduction.** The personnel-assignment problem can be formulated as follows: A certain company needs to fill  $N$  jobs, and  $N$  applicants are available. The value (i.e., the net profit to the company) of each applicant depends upon the job in which he is placed. Assuming these values to be known, the problem is to assign the applicants in such a way that the total value is a maximum.

Frequently there are many identical jobs which demand the same qualifications. Such jobs can be combined into a job category.

In this paper we give a very simple combinatorial method to solve the assignment problem with three such job categories. This method is very efficient even for big  $N$ , the maximum number of operations being roughly  $KN \log N$ , where  $K$  is independent of  $N$ .

Let  $b_j$  denote the number of jobs grouped in the  $j^{\text{th}}$  category,  $v_{ij}$  the values of the  $i^{\text{th}}$  applicant when placed in the  $j^{\text{th}}$  category ( $i = 1, 2, \dots, N$ ); ( $j = 1, 2, 3$ ). Thus  $N = b_1 + b_2 + b_3$ .

**Definitions.** An assignment  $s$  is a function which assigns the  $i^{\text{th}}$  applicant to the  $s(i)^{\text{th}}$  job category,  $i = 1, 2, \dots, N$ ,  $1 \leq s(i) \leq 3$ . A feasible assignment in our problem is an assignment which assigns exactly  $b_j$  applicants to the  $j^{\text{th}}$  category,  $j = 1, 2, 3$ . The total value of a feasible assignment  $s$  is

$$v(s) = \sum_{i=1}^N v_{i,s(i)}$$

Our assignment problem is, of course, to find a feasible assignment the total value of which is a maximum. There may be, however, several such maximum assignments. To avoid ambiguity we introduce the following "perturbed" values:

$$v_{ij}^e = v_{ij} + j\epsilon^i, \quad v^e(s) = \sum_{i=1}^N v_{i,s(i)}^e.$$

It is easy to see that there is one and only one feasible assignment  $s_0$  whose perturbed-

total-value is a maximum for all sufficiently small positive  $\varepsilon$ , i.e.  $v^\varepsilon(s_0) > v^\varepsilon(s)$  for all feasible solutions  $s \neq s_0$  and all  $0 < \varepsilon < \varepsilon_0$ . We call  $s_0$  the maximum assignment. It is our purpose to compute  $s_0$ .

**2. The solution of the Problem.** For fixed  $k$  and  $l$  ( $1 \leq k \neq l \leq 3$ ), we compute the differences

$$\Delta_{kl}(i) = v_{ik} - v_{il}, \quad i = 1, 2, \dots, N,$$

and arrange the  $N$  applicants in a sequence of decreasing  $\Delta_{kl}(i)$ . Let  $p_{kl}(i)$  denote the place of the  $i^{\text{th}}$  applicant in that sequence. Thus  $p_{kl}(i)$  is a permutation such that  $p_{kl}(i_1) < p_{kl}(i_2)$  if and only if

$$\begin{aligned} &\text{either } \Delta_{kl}(i_1) > \Delta_{kl}(i_2) \\ &\text{or } \Delta_{kl}(i_1) = \Delta_{kl}(i_2) \quad \text{and} \quad (k-l)(i_2 - i_1) > 0. \end{aligned}$$

Note that these rules are designed to ensure strictly decreasing order in the perturbed differences

$$\Delta_{kl}^\varepsilon(i) = v_{ik}^\varepsilon - v_{il}^\varepsilon = \Delta_{kl}(i) + (k-l)\varepsilon^i, \quad (0 < \varepsilon < \varepsilon_0).$$

Also note that  $p_{kl}$  is the reverse of  $p_{lk}$ , namely,  $p_{kl}(i) = N + 1 - p_{lk}(i)$ .

**Proposition 1.** *If  $p_{kl}(i) \leq b_k$  then  $s_0(i) \neq l$ .*

In other words, the first  $b_k$  applicants in the sequence cannot be assigned, in the maximum assignment, to the  $l^{\text{th}}$  category. We shall say that they are labeled as non- $l$  workers.

**Proof.** Suppose  $s_0(i) = l$ . Then there must exist  $i_1$  such that  $s_0(i_1) = k$  and  $p_{kl}(i_1) > b_k$ . Let  $s_1$  be an assignment identical with  $s_0$  except for  $s_1(i) = k$  and  $s_1(i_1) = l$ . Thus

$$v^\varepsilon(s_1) = v^\varepsilon(s_0) + \Delta_{kl}^\varepsilon(i) - \Delta_{kl}^\varepsilon(i_1).$$

But  $p_{kl}(i_1) > p_{kl}(i)$  and therefore  $\Delta_{kl}^\varepsilon(i) > \Delta_{kl}^\varepsilon(i_1)$ . Hence  $v^\varepsilon(s_1) > v^\varepsilon(s_0)$ , which contradicts our definition of the maximum assignment  $s_0$ . Therefore we must have  $s_0(i) \neq l$ .

Similarly we can show that if  $p_{kl}(i) > N - b_l$  then  $s_0(i) \neq k$ . Thus the last  $b_l$  applicants in the  $p_{kl}$  sequence are labeled as non- $k$  workers.

Our algorithm proceeds as follows: We write down the three sequences (permutations)  $p_{12}$ ,  $p_{23}$  and  $p_{31}$ . The first  $b_k$  and the last  $b_l$  applicants of each sequence  $p_{kl}$  are labeled as non- $l$  and non- $k$  workers, respectively. Since each applicant appears in all three sequences, many of them are likely to get two different labels and thus be conclusively assigned. For example, if  $p_{12}(i) \leq b_1$  and  $p_{31}(i) > N - b_1$  then by the

above proposition  $s_0(i) \neq 2$  and  $s_0(i) \neq 3$ , and hence  $s_0(i) = 1$ . The  $i^{\text{th}}$  applicant is then assigned to the first job category, and the problem dimension  $N$  is reduced.

In this way we can continue to mark workers off until the problem is either completely solved or reduced to a problem where no applicant has two different labels. This reduced problem is immediately solved by Propositions 2 and 3 below.

**Proposition 2.** *If no applicant has two different labels then  $b_1 = b_2 = b_3$  and every applicant has exactly one label.*

**Proof.** With no loss of generality we may assume that  $b_1 \geq b_2 \geq b_3$ . Let  $v \geq 0$  be the number of applicants with no label. Using Proposition 1 we get at least  $b_2$  applicants with non-1 labels, at least  $b_1$  with non-2 labels and at least  $b_1$  with non-3 labels. Thus the total number of different labels is at least  $2b_1 + b_2$ , i.e.,

$$b_1 + b_2 + b_3 - v \geq 2b_1 + b_2 .$$

This implies  $b_3 \geq b_1 + v \geq b_3 + v$ , which in turn yields  $v = 0$  and  $b_1 = b_3$ . Q.E.D.

Thus, in the reduced problem we have  $N = 3b$ ,  $b_j = b$ , and, for a fixed  $1 \leq j \leq 3$ , exactly  $b$  applicants are labeled as non- $j$  workers. Let these  $b$  applicants be ordered in a sequence of decreasing  $\Delta_{kl}$  (strictly decreasing  $\Delta_{kl}^e$ . ( $j, k, l$ ) here is any cyclic permutation of (1, 2, 3)), and denote by  $i_{kl}(p)$  the index of the applicant that appears in the  $p^{\text{th}}$  place of this sequence. In fact this ordering requires no extra effort, for, by Proposition 1-2, all the non- $j$  labeled applicants appear at the places  $b + 1$ ,  $b + 2$ , ...,  $2b$  in the sequence  $p_{kl}$ , where they are ordered in decreasing  $\Delta_{kl}$  (strictly decreasing  $\Delta_{kl}^e$ ); i.e.,

$$i_{kl}(p_{kl}(i) - b) = i, \quad b < p_{kl}(i) \leq 2b .$$

Now, clearly, if  $s_0(i_{kl}(p_0)) = k$  then  $s_0(i_{kl}(p)) = k$  for all  $p < p_0$ . For if  $s_0(i_{kl}(p)) = l$  we could define another assignment,  $s_1$  identical with  $s_0$  except for  $s_1(i_{kl}(p_0)) = l$  and  $s_1(i_{kl}(p)) = k$ , for which we would have

$$v^e(s_1) = v^e(s_0) - \Delta_{kl}^e(i_{kl}(p_0)) + \Delta_{kl}^e(i_{kl}(p)) > v^e(s_0) ,$$

in contradiction to the definition of  $s_0$ . Therefore there must be a certain place,  $q_{kl}$  say ( $0 \leq q_{kl} \leq b$ ), such that

$$s_0(i_{kl}(p)) = \begin{cases} k & \text{for } 1 \leq p \leq q_{kl} \\ l & \text{for } q_{kl} < p \leq b . \end{cases}$$

For this assignment to be feasible there must be exactly  $b$  applicants assigned to the  $k^{\text{th}}$  category, that is,

$$q_{kl} + (b - q_{jk}) = b .$$

This entails  $q_{12} = q_{23} = q_{31} = q_0$  (say), and therefore  $s_0 = s^{q_0}$ , where the assignments  $s^q$  ( $0 \leq q \leq b$ ) are defined by

$$s^q(i_{kl}(p)) = \begin{cases} k & \text{for } 1 \leq p \leq q, \\ l & \text{for } q < p \leq b, \end{cases} \quad (k, l) = (1, 2), (2, 3), (3, 1).$$

All we need to conclude our solution is to determine  $q_0$ . This is done by observing that

$$v^\varepsilon(s^q) - v^\varepsilon(s^{q-1}) = C^\varepsilon(q)$$

where

$$\begin{aligned} C^\varepsilon(q) &= \Delta_{12}^\varepsilon(i_{12}(q)) + \Delta_{23}^\varepsilon(i_{23}(q)) + \Delta_{31}^\varepsilon(i_{31}(q)) = \\ &= C(q) - \varepsilon^{i_{12}(q)} - \varepsilon^{i_{23}(q)} + 2\varepsilon^{i_{31}(q)}, \\ C(q) &= \Delta_{12}(i_{12}(q)) + \Delta_{23}(i_{23}(q)) + \Delta_{31}(i_{31}(q)). \end{aligned}$$

Since  $\Delta_{ki}^\varepsilon(i_{ki}(q))$  is a strictly decreasing function of  $q$ , so is also  $C^\varepsilon(q)$  and hence  $q_0$  must be the last  $q$  for which  $C^\varepsilon(q) > 0$  for all sufficiently small positive  $\varepsilon$ . We have thus proved

**Proposition 3.** *Let  $q_0$  be the last  $q$  for which*

*either  $C(q) > 0$*

*or  $C(q) = 0$ ,  $i_{31}(q) < i_{12}(q)$  and  $i_{31}(q) < i_{23}(q)$ .*

*(Put  $q_0 = 0$  if no  $q$  satisfies this condition.) Then  $s_0 = s^{q_0}$ .*

This proposition gives us the solution of the reduced problem (where no applicant has two different labels) and thus conclude the solution of the problem.

**Number of operations.** The above algorithm can be programmed so that the assignment of each applicant takes no more than a fixed number  $k$  of operations,  $k$  being independent of  $N$ . To this one has to add the three sortings needed initially to set up the permutations  $P_{12}(i)$ ,  $P_{23}(i)$  and  $P_{31}(i)$ . Each such sorting can be executed in at most  $KN \log N$  operations,  $K$  being independent of  $N$ . Thus, for big  $N$ , the initial sortings constitute the most time consuming part of the algorithm, and the total number of operations is  $O(N \log N)$ .

*Authors' address:* The Weizmann Institute of Science, Rehovot, Israel.