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ORDER OF HOLONOMY OF A SURFACE WITH PROJECTIVE CONNECTION

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A submanifold in a space with Cartan connection, see [3], represents a natural generalization of a submanifold in the corresponding homogeneous space. É. CARTAN himself showed in the case of a surface in a 3-dimensional space with projective connection, [1], that his method of specialization of frames can also be applied to the investigation of these submanifolds. A. ŠVEC pointed out, cf. [5], that such a submanifold can be considered as a separate structure. From this point of view, a surface in a 3-space with projective connection is called a manifold of type $P^2_{0,3}$, or, shortly, a surface with projective connection. Naturally, differential geometry of a surface $\mathcal{P}$ with projective connection differs from differential geometry of a surface in projective 3-space P_3. In this paper, we want to show that the difference between $\mathcal{P}$ and a surface in P_3 can be also measured in individual orders. If we use the computational procedures by É. Cartan, then the difference in order k between $\mathcal{P}$ and a surface in P_3 is characterized by the difference between the formulae of the $(k - 1)$-st prolongation for $\mathcal{P}$ and the formulae of the $(k - 1)$-st prolongation for a surface in P_3. Conversely, if these formulae coincide, then we say that $\mathcal{P}$ is holonomic of order k, or, shortly, k-holonomic. Dealing with the first prolongation, we show the invariance of the condition for 2-holonomy also in a formal computional way, but we do not repeat it for higher orders, since we present a direct invariant definition of k-holonomy for an arbitrary manifold with connection in [4].

At every order, we geometrize the corresponding conditions for holonomy by means of some properties of some geometric objects of $\mathcal{P}$. In general, the geometric objects of $\mathcal{P}$ differ from the geometric objects of a surface in P_3. But if $\mathcal{P}$ is k-holonomic and if we take into account how one evaluates the geometric objects of order k of $\mathcal{P}$, then we are led to the following proposition: $\mathcal{P}$ is k-holonomic if and only if all its geometric objects of order k are analogous to geometric objects of order k of a surface in P_3. We present an exact formulation of this assertion for an arbitrary manifold with connection as well as its proof in [4]. Our considerations end at the sixth order, since a 6-holonomic non-special surface with projective connection has integrable connection, so that it is locally isomorphic to a surface in P_3 and is holonomic of any order. The totality of our geometric conditions gives a necessary and sufficient geometric condition that a surface with projective connection be locally equivalent to a surface in P_3, which is a problem solved by B. CENKL, [2]. In contradistinction to this paper, our conditions are organized according to individual orders.

1. Consider a surface $\mathscr{P}$ with projective connection together with the manifold $\mathscr{F}_{12}$ of all frames associated with $\mathscr{P}$, which depend on 12 secondary parameters. Let the connection be given by

$$\mathrm{d}A_i = \omega_i^j A_j \tag{1}$$

where ω_j^i are differential forms on $\mathscr{F}_{12}$ satisfying

$$\omega_i^i = 0\,. \tag{2}$$

The structure equations are

$$\mathrm{d}\omega_i^j = \omega_i^k \wedge \omega_k^j + 2R_i^j\omega^1 \wedge \omega^2 \tag{3}$$

and it holds

$$R_i^i = 0\,. \tag{4}$$

(We write $\omega_0^1 = \omega^1$, $\omega_0^2 = \omega^2$ as usual.)

2. The frame field $\mathscr{F}_{10}$ of the first order is determined by the usual relation

$$\omega_0^3 = 0\,. \tag{5}$$

The exterior differentiation of (5) yields

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 + 2R_0^3\omega^1 \wedge \omega^2 = 0\,, \tag{6}$$

which is equivalent to

$$\omega_1^3 = a_1\omega^1 + (a_2 - R_0^3)\,\omega^2\,, \quad \omega_2^3 = (a_2 + R_0^3)\,\omega^1 + a_3\omega^2\,. \tag{7}$$

Prolonging (7) and fixing the principal parameters, we obtain

$$\begin{aligned}
&\delta a_1 + a_1(e_0^0 - 2e_1^1 + e_3^3) - 2a_2e_1^2 = 0\,,\\
&\delta a_2 + a_2(e_0^0 - e_1^1 - e_2^2 + e_3^3) - a_1e_2^1 - a_3e_1^2 = 0\,,\\
&\delta a_3 + a_3(e_0^0 - 2e_2^2 + e_3^3) - 2a_2e_2^1 = 0\,,\\
&\delta R_0^3 + R_0^3(e_0^0 - e_1^1 - e_2^2 + e_3^3) = 0\,,
\end{aligned} \tag{8}$$

so that R_0^3 is a relative invariant. If it holds

$$R_0^3 = 0\,, \tag{9}$$

then $\mathscr{P}$ will be called 2-*holonomic*. Furthermore, we can deduce from (8) that the quantity

$$h = \frac{R_0^3}{\sqrt{[(a_2)^2 - a_1 a_3]}} \tag{10}$$

is an absolute invariant.

Restricting ourselves to the investigation of hyperbolic surfaces, we can specialize the frames by $a_2 = 1, a_1 = a_3 = 0$ and we get the frame field $\mathscr{F}_7$ of the second order. When comparing with [6], we find that h is the torsion of $\mathscr{P}$ and we have deduced the following geometric assertion: *$\mathscr{P}$ is 2-holonomic at a point if and only if the conjugate tangents at this point form an involution.*

3. From now on, we shall suppose $\mathscr{P}$ is 2-holonomic, so that we have

$$\omega_1^3 = \omega^2\,, \quad \omega_2^3 = \omega^1 \tag{11}$$

and the prolongation of (11) yields

$$\begin{aligned} 2\omega_1^2 &= b_1\omega^1 + (b_2 - R_0^2 + R_1^3)\,\omega^2\,, \\ -\omega_0^0 + \omega_1^1 + \omega_2^2 - \omega_3^3 &= (b_2 + R_0^2 - R_1^3)\,\omega^1 + (b_3 - R_0^1 + R_2^3)\,\omega^2\,, \\ 2\omega_2^1 &= (b_3 + R_0^1 - R_2^3)\,\omega^1 + b_4\omega^2\,. \end{aligned} \tag{12}$$

If it holds

$$R_1^3 = R_0^2\,, \quad R_2^3 = R_0^1\,, \tag{13}$$

then $\mathscr{P}$ will be said to be 3-*holonomic*.

Now we give a geometric interpretation of (13). Consider the ruled surface $\mathscr{L}_1$ generated by the tangent lines to the asymptotic curves $\omega^2 = 0$ along an asymptotic curve $\omega^1 = 0$ as well as the ruled surface $\mathscr{L}_2$ generated symmetrically. It is easy to see that the quadrics having the first order (line) contact with both $\mathscr{L}_1$ and $\mathscr{L}_2$ form the pencil

$$2x^0x^3 - 2x^1x^2 + (b_2 - R_0^2 + R_1^3)\,x^1x^3 + (b_3 + R_0^1 - R_2^3)\,x^2x^3 = a_{33}(x^3)^2 \tag{14}$$

where x^0, x^1, x^2, x^3 are the local coordinates. On the other hand, consider a quadric Q having the second order contact with $\mathscr{P}$, then there are exactly three directions in which $\mathscr{P}$ has the third order contact with Q. These directions are apolar with respect to the asymptotic directions if and only if Q belongs to the following pencil

$$\begin{aligned} 2x^0x^3 - 2x^1x^2 + [b_2 + \tfrac{1}{3}(R_0^2 - R_1^3)]\,x^1x^3 &+ \\ + [b_3 - \tfrac{1}{3}(R_0^1 - R_2^3)]\,x^2x^3 &= \bar{a}_{33}(x^3)^2\,, \end{aligned} \tag{15}$$

cf. [6], p. 389. Comparing (13), (14), (15), we can conclude: *A 2-holonomic surface $\mathscr{P}$ is 3-holonomic if and only if both preceding constructions give the same pencil of quadrics (of Darboux).*

4. In what follows, $\mathscr{P}$ will be supposed to be 3-holonomic and non-ruled. Standard procedure shows that we can further specialize the frames by $b_1 = b_4 = 2$, $b_2 = b_3 = 0$ and we get the frame field $\mathscr{F}_3$ of the third order. Prolonging the equations

$$\omega_1^2 = \omega^1\,, \quad \omega_2^1 = \omega^2\,, \quad \omega_1^1 + \omega_2^2 = 0\,, \quad \omega_0^0 + \omega_3^3 = 0\,, \tag{16}$$

we obtain

$$\begin{aligned} \omega_0^0 - 3\omega_1^1 &= c_1\omega^1 + (c_2 + R_0^1 - R_1^2)\,\omega^2\,, \\ \omega_3^2 - \omega_1^0 &= (c_2 - R_0^1 + R_1^2)\,\omega^1 + (c_3 - R_1^1 - R_2^2)\,\omega^2\,, \\ \omega_3^1 - \omega_2^0 &= (c_3 + R_1^1 + R_2^2)\,\omega^1 + (c_4 + R_0^2 - R_2^1)\,\omega^2\,, \\ \omega_0^0 + 3\omega_1^1 &= (c_4 - R_0^2 + R_2^1)\,\omega^1 + c_5\omega^2\,. \end{aligned} \tag{17}$$

If it holds

$$R_1^2 = R_0^1\,, \quad R_2^1 = R_0^2\,, \quad R_1^1 + R_2^2 = 0\,, \tag{18}$$

then $\mathscr{P}$ will be called *4-holonomic*.

The osculating quadric of the ruled surface $\mathscr{L}_1$ or $\mathscr{L}_2$ considered in item 3 has the equation

$$2x^0x^3 - 2x^1x^2 + (c_3 - R_1^1 - R_2^2)(x^3)^2 = 0 \tag{19}$$

or

$$2x^0x^3 - 2x^1x^2 + (c_3 + R_1^1 + R_2^2)(x^3)^2 = 0 \tag{20}$$

respectively, so that both quadrics coincide if and only if $R_1^1 + R_2^2 = 0$. In the sequel we suppose that this condition holds and (19) = (20) will be called the quadric of Lie.

Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two line congruences associated with $\mathscr{P}$ in such a way that the lines of $\mathscr{K}_1$ pass through the corresponding point of $\mathscr{P}$ but do not lie in the tangent plane and the lines of $\mathscr{K}_2$ lie in the corresponding tangent plane of $\mathscr{P}$ but do not pass through the point of contact. Then $\mathscr{K}_1$ and $\mathscr{K}_2$ are said to be reciprocal, if their lines are conjugate with respect to the quadric of Lie. If $\mathscr{K}_1$ is generated by the straight line $[A_0, pA_1 + qA_2 + A_3]$, then the reciprocal $\mathscr{K}_2$ is generated by $[qA_0 + A_1, pA_0 + A_2]$ and the focal nets of both congruences coincide if and only if

$$p = -\tfrac{1}{4}(c_2 - R_0^1 + R_2^2)\,, \quad q = -\tfrac{1}{4}(c_4 + R_0^2 - R_2^1)\,; \tag{21}$$

these lines will be called the first or the second directrix of Wilczynski respectively.

By the principal quadrics of $\mathscr{P}$ we mean those quadrics which have contact of the fourth order with the asymptotic curves of $\mathscr{P}$; they form the following pencil

$$(22) \qquad 2x^1x^2 - 6x^0x^3 + c_1x^1x^3 + c_5x^2x^3 = a_{33}(x^3)^2 .$$

There exists exactly one pair of reciprocal congruences whose lines are also conjugate with respect to (22); these lines will be called the edges of Green. The first edge of Green is

$$(23) \qquad [A_0, \tfrac{1}{4}c_5A_1 + \tfrac{1}{4}c_1A_2 + A_3] .$$

The curves of Segre are given by $(\omega^1)^3 - (\omega^2)^3 = 0$. The first axis of Čech is the common line of intersection of the osculating planes of the curves of Segre, which is

$$(24) \qquad [A_0, \tfrac{1}{6}(c_5 - c_2 - R_0^1 + R_1^2) A_1 + \tfrac{1}{6}(c_1 - c_4 + R_0^2 - R_2^1) A_2 + A_3] .$$

The lines (21), (23), (24) belong to the same (canonical) pencil if and only if $R_1^2 = R_0^1$, $R_2^1 = R_0^2$. Thus, *a 3-holonomic surface $\mathscr{P}$ is 4-holonomic if and only if the osculating quadrics of ruled surfaces $\mathscr{L}_1$ and $\mathscr{L}_2$ coincide and if the directrix of Wilczynski, the edge of Green and the axis of Čech belong to the same pencil.*

5. Suppose $\mathscr{P}$ is 4-holonomic. The remaining secondary parameters can be fixed by $c_2 = c_3 = c_4 = 0$ and we get the canonical frame field $\mathscr{F}$. Then we have

$$(25) \qquad \omega_0^0 - 3\omega_1^1 = c_1\omega^1 , \quad \omega_3^2 = \omega_1^0 , \quad \omega_3^1 = \omega_2^0 , \quad \omega_0^0 + 3\omega_1^1 = c_5\omega^2 .$$

Prolonging (25), we obtain

$$(26) \qquad \begin{aligned} -dc_1 - c_1(\omega_0^0 - \omega_1^1) - 4\omega_1^0 + 3\omega^2 &= e_1\omega^1 + (e_2 + R_0^0 - 3R_1^1 - c_1R_0^1)\,\omega^2 , \\ 2\omega_2^0 &= (e_2 - R_0^0 + 3R_1^1 + c_1R_0^1)\,\omega^1 + (e_3 + R_3^2 - R_1^0)\,\omega^2 , \\ 2\omega_3^0 &= (e_3 - R_3^2 + R_1^0)\,\omega^1 + (e_4 + R_3^1 - R_2^0)\,\omega^2 , \\ 2\omega_1^0 &= (e_4 - R_3^1 + R_2^0)\,\omega^1 + (e_5 + R_0^0 + 3R_1^1 - c_5R_0^2)\,\omega^2 , \\ -dc_5 - c_5(\omega_0^0 + \omega_1^1) - 4\omega_2^0 + 3\omega^1 &= (e_5 - R_0^0 - 3R_1^1 + c_5R_0^2)\,\omega^1 + e_6\omega^2 . \end{aligned}$$

If it holds

$$(27) \qquad R_0^0 - 3R_1^1 = c_1R_0^1 , \quad R_3^2 = R_1^0 , \quad R_3^1 = R_2^0 , \quad R_0^0 + 3R_1^1 = c_5R_0^2 ,$$

then $\mathscr{P}$ will be called 5-*holonomic*.

The first normal of Fubini is the line harmonically conjugate to the canonical tangent with respect to the directrix of Wilczynski and the edge of Green, which is

$$(28) \qquad [A_0, \tfrac{1}{2}c_5A_1 + \tfrac{1}{2}c_1A_2 + A_3] .$$

The developable surfaces of this congruence intersect a conjugate net on $\mathcal{P}$ if and only if

$$c_1 R_0^1 + c_5 R_0^2 - 2R_0^0 = 0\,. \tag{29}$$

The second focal surface of the congruence of the tangents to a family of curves of Segre is without torsion if and only if

$$c_1 R_0^1 - c_5 R_0^2 + 6R_1^1 = 0\,. \tag{30}$$

Furthermore, consider the envelope of the quadrics of Lie

$$x^1 x^2 - x^0 x^3 = 0\,. \tag{31}$$

It is easy to see that the characteristic points, i.e. the vertices of the tetrahedron of Demoulin, are determined by (31) and by

$$a_3^0 (x^3)^2 - (x^1)^2 = 0\,, \quad b_3^0 (x^3)^2 - (x^2)^2 = 0\,. \tag{32}$$

The transversals of the tetrahedron of Demoulin intersect the lines $[A_0 A_3]$ and $[A_1 A_2]$ at

$$\begin{aligned} D_1 &= (\sqrt{a_3^0 b_3^0}, 0, 0, 1)\,, \quad D_3 = (0, \sqrt{a_3^0}, \sqrt{b_3^0}, 0) \\ D_2 &= (-\sqrt{a_3^0 b_3^0}, 0, 0, 1)\,, \quad D_4 = (0, \sqrt{a_3^0}, -\sqrt{b_3^0}, 0)\,, \end{aligned} \tag{33}$$

where the pairs D_1, D_3 and D_2, D_4 lie on the same transversal. If $\xi A_0 + \eta A_3$ or $\lambda A_1 + \mu A_2$ are the coordinates on $[A_0 A_3]$ or $[A_1 A_2]$, then the pair D_1, D_2 or D_3, D_4 has the equation

$$\xi^2 - a_3^0 b_3^0 \eta^2 = 0 \tag{34}$$

or

$$b_3^0 \lambda^2 - a_3^0 \mu^2 = 0 \tag{35}$$

respectively. On the other hand, the focal planes of the congruence of the first directrices of Wilczynski intersect $[A_1 A_2]$ at the points

$$a_1^0 \lambda^2 + (b_1^0 - a_2^0)\, \lambda\mu - b_2^0 \mu^2 = 0\,. \tag{36}$$

Thus, *the pairs* (35), (36) *and* A_1, A_2 *belong to the same involution* if and only if

$$a_1^0 a_3^0 = b_2^0 b_3^0\,. \tag{37}$$

The foci of the congruence of the first directrices of Wilczynski are determined by

$$\xi^2 + \xi\eta(a_2^0 + b_1^0) + \eta^2(a_2^0 b_1^0 - a_1^0 b_2^0) = 0\,. \tag{38}$$

The tangent plane of the surface (A_1) or (A_2) intersects $[A_0A_3]$ at $T_1 = b_1^0A_0 + A_3$ or $T_2 = a_2^0A_0 + A_3$ respectively. Let T_3 be the harmonically conjugate of A_0 with respect to T_1, T_2, let T_4 be the harmonically conjugate of A_3 with respect to A_0, T_3 and let T be the harmonically conjugate of T_4 with respect to A_0, A_3, then the pair A_0, T is given by

(39) $$(a_2^0 + b_1^0)\,\xi\eta + a_2^0b_1^0\eta^2 = 0\,.$$

The pairs (34), (38), (39) *belong to the same involution* if and only if

(40) $$a_1^0b_2^0 = a_3^0b_3^0\,.$$

(37) and (40) imply

(41) $$a_3^0 = \varepsilon b_2^0\,,\quad b_3^0 = \varepsilon a_1^0\,,\quad \varepsilon = \pm 1\,.$$

On the other hand, the relations $R_3^2 = R_1^0$, $R_3^1 = R_2^0$ are equivalent to $b_2^0 = a_3^0$, $b_3^0 = a_1^0$, cf. (26). The additional condition $\varepsilon = 1$ for (41) is equivalent to *the following condition concerning orientation.* Let F_1, F_2 be the foci of the congruence of the first directrices of Wilczynski taken in such order that the orientation on $[A_0A_3]$ determined by the ordered triple (A_0, F_1, F_2) coincides with the orientation (A_0, D_1, D_2). Let F_{i+2}, $i = 1, 2$, be the point of intersection of the focal plane passing through F_i with $[A_1A_2]$. Then the orientation (A_1, F_3, F_4) coincides with the orientation (A_1, D_3, D_4) if and only if $\operatorname{sgn} a_3^0 = \operatorname{sgn} b_2^0$, i.e. $\varepsilon = 1$. – Thus *we have deduced necessary and sufficient geometric conditions that a 4-holonomic surface $\mathcal{P}$ be 5-holonomic.*

6. Suppose $\mathcal{P}$ is 5-holonomic. Analogous considerations as above suggest the following definition. If it holds

$$-c_1R_0^0 + c_1R_1^1 - 4R_1^0 + 3R_0^2 = e_1R_0^1 + e_2R_0^2\,,$$

(42) $$2R_2^0 = e_2R_0^1 + e_3R_0^2\,,\quad 2R_3^0 = e_3R_0^1 + e_4R_0^2\,,\quad 2R_1^0 = e_4R_0^1 + e_5R_0^2\,,$$

$$-c_5R_0^0 - c_5R_1^1 - 4R_2^0 + 3R_0^1 = e_5R_0^1 + e_6R_0^2\,,$$

then $\mathcal{P}$ will be said to be *6-holonomic*.

It is easy to see that

(43) $$2R_3^0 = e_3R_0^1 + e_4R_0^2$$

holds if and only if *the surface (A_3) is without torsion* and that

(44) $$2R_0^2 = e_2R_0^1 + e_3R_0^2\,,\quad 2R_1^0 = e_4R_0^1 + e_5R_0^2$$

are satisfied if and only if *both focal surfaces of the congruence of the first directrices of Wilczynski are without torsion.* Now, taking $(42_{1,5})$ modulo $(27_{1,4})$, (43), (44), we obtain two linear homogeneous equations in R_0^1, R_0^2. Since R_0^1, R_0^2 are the last independent components of the curvature tensor, there are many possibilities how to geometrize these equations; we choose the simpliest way: If the determinant D of this system does not vanish, then $(43_{1,5})$ holds if and only if $R_0^1 = R_0^2 = 0$, i.e. if *the torsion tensor of* $\mathcal{P}$ *vanishes.*

7. Summarizing the preceding considerations, we get the following result, which concludes our investigation in general case. *If a non-ruled surface* $\mathcal{P}$ *with* $D \neq 0$ *is 6-holonomic, then its curvature tensor vanishes, so that its connection is integrable.*

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