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*Časopis pro pěstování matematiky*, Vol. 98 (1973), No. 1, 1--21

Persistent URL: `http://dml.cz/dmlcz/117779`

ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY
*Vydává Matematický ústav ČSAV, Praha*
SVAZEK 98 * PRAHA 14. 2. 1973 * ČÍSLO 1

# ON THE ZEROS OF GENERALIZED JACOBI'S ORTHOGONAL POLYNOMIALS

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(Received December 15, 1970)

## 1. INTRODUCTION

**1,1.** We employ the following notation:

1. $I$ is the closed interval $[-1, 1]$.

2. $c_i$ $(i = 1, 2, \ldots)$ are positive constants independent of $n$ as well as of $x \in I$ or of $x$ in the interval in question.

3. $c_i(x)$ $(i = 1, 2, \ldots)$ are functions of the variable $x$ such that

$$|c_i(x)| < c_1 .$$

The numbering of $c_i$ a $c_i(x)$ is independent for every section.

**1,2.** In this paper the zeros of the orthonormal polynomials

$$Q_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k} , \quad a_0^{(n)} > 0 , \quad n = 0, 1, \ldots \tag{1,2a}$$

associated with the function

$$Q(x) = (1 - x)^\alpha (1 + x)^\beta e^{u(x)} = J(x) . e^{u(x)} \tag{1,2b}$$

on the interval $I$ are investigated. Here $\alpha > -1$, $\beta > -1$ and $u(x)$ is a real function satisfying the following conditions:

1. $u'''(x)$ exists in the interval $[-1, 1]$.

2. If we put for brevity

$$\Delta_x f(t) = \frac{f(x) - f(t)}{x - t}, \quad v_1(t) = \Delta_x u''(t), \quad v_2(t) = \frac{\partial^2}{\partial x\, \partial t} \Delta_x u'(t), \quad v_3(t) = \Delta_x u'''(t),$$

then for $i = 1, 2, 3$

$$(1{,}2c) \qquad \min(\alpha, \beta) \geqq \tfrac{1}{2} \Rightarrow \int_I (t - t^2)^{3/2} |v_i(t)| \, dt = c_1(x)$$

and

$$(1{,}2d) \qquad \min(\alpha, \beta) < \tfrac{1}{2} \Rightarrow \Delta_x v_i(t) = c_2(x).$$

**1,3.** In my paper "On a class of generalized Jacobi's orthonormal polynomials"[1]) I have established the following differential equation for the above polynomials $Q_n(x)$:

(1,3a)

$$Q^{-1}(x) \frac{d}{dx} [(1 - x^2) Q_n'(x) Q(x) + (1 - x^2) b_n(x) Q_n'(x) + [\lambda_n^2 + a_n(x)] Q_n(x) = 0.$$

Herein

$$(1{,}3b) \qquad \lambda_n = \sqrt{(n(n + \alpha + \beta + 1))}$$

(We suppose $n$ to be so large that $\lambda_n$ is real.)

Further

$$(1{,}3c) \qquad a_n(x) = n\, c_3(x),$$

$$(1{,}3d) \qquad b_n(x) = n^{-1} c_4(x),$$

$b_n'(x)$ exists in the interval $[-1, 1]$ and

$$(1{,}3e) \qquad b_n'(x) = n^{-1} c_5(x).$$

**1,4.** We denote by $J_n(x)$ the orthonormal polynomial associated with the function $J(x)$ on the interval $[-1, 1]$. $J_n(x)$ are normalized Jacobi's polynomials.

**1,5.** The results of my investigations are contained in the second chapter. The theorems on the zeros of the polynomials $J_n(x)$ are a generalization of the known results of Szegö (See [7] p. 9 and [1] pp. 135–136).

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[1]) See Čas. pěst. mat. *97* (1972), 361–378.

## 2. THEOREMS ON THE ZEROS OF THE POLYNOMIALS $Q_n(x)$

**2,1.** *Let $\{x_{\nu,n}\}_{n=1}^{\infty}$ be the increasing sequence of the zeros of Bessel function $I_\nu(x)$ of the first kind and of order $\nu$.*

*Let $\{x_k^{(n)}\}_{k=1}^{n}$ be the increasing sequence of zeros of the polynomial $Q_n(x)$.*

*Let $k = 1, 2, \ldots$ be independent of $n$. Then for $n \to +\infty$*

(2,1a) $$x_k^{(n)} = -1 + \frac{x_{\beta,k}^2}{2n^2}\left[1 + O(n^{-1})\right]$$

and

(2,1b) $$x_{m-k+1}^{(n)} = 1 - \frac{x_{\alpha,k}^2}{2n^2}\left[1 + O(n^{-1})\right].$$

(*The constants in $O$ depend on $k$.*)

The proof of this theorem is contained in Chapter 5.

**2,2.** *Let $Q_n(x) = J_n(x)$ where $J_n(x)$ is defined in Section* 1,4. *If we put*

(2,2a) $$j(\alpha, \beta) = j = \tfrac{1}{6}(\alpha^2 + 3\alpha\beta + 3\alpha + 3\beta + 2)\,, \quad j_1 = j(\beta, \alpha)\,,$$

*then*

(2,2b) $$x_k^{(n)} = -1 + \frac{x_{\beta,k}^2}{2n^2}\left[1 - \frac{\alpha + \beta + 1}{n} - \frac{(\alpha + \beta + 1)^2 + j_1}{n^2} - \right.$$
$$\left. - \frac{(\alpha + \beta + 1)\left[2j_1 + (\alpha + \beta + 1)^2\right]}{n^3}\right] - \frac{x_{\beta,k}^4}{24n^4}\left[1 - \frac{2(\alpha + \beta + 1}{n}\right] + O(n^{-6})$$

*and*

(2,2c) $$x_{n-k+1}^{(n)} = 1 - \frac{x_{\alpha,k}^2}{2n^2}\left[1 - \frac{\alpha + \beta + 1}{n} - \frac{(\alpha + \beta + 1)^2 + j}{n^2} - \right.$$
$$\left. - \frac{(\alpha + \beta + 1)\left[2j + (\alpha + \beta + 1)^2\right]}{n^3}\right] + \frac{x_{\alpha,k}^4}{24n^4}\left[1 - \frac{2(\alpha + \beta + 1)}{n}\right] + O(n^{-6})\,.$$

The proof is in Chapter 6.

**2,3. Theorem on the distance of the consecutive zeros of the function $Q_n(\sin z)$.**

Notations.

(2,3a) $$|\alpha| \leqq \tfrac{1}{2} \Rightarrow \alpha_1 = 0\,; \quad |\alpha| > \tfrac{1}{2} \Rightarrow \alpha_1 = \tfrac{1}{2}\sqrt{(4\alpha^2 - 1)}\,;$$

(2,3b) $$|\beta| \leqq \tfrac{1}{2} \Rightarrow \beta_1 = 0\,; \quad |\beta| > \tfrac{1}{2} \Rightarrow \beta_1 = -\tfrac{1}{2}\sqrt{(4\beta^2 - 1)}\,.$$

$\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$ are arbitrary real numbers independent of $n$;

(2,3c) $$a_n \in (\alpha_0, n), \quad b_n \in (-n, \beta_0)$$

are arbitrary numbers which may depend on $n$;

(2,3d) $$J_n = \left(-\frac{\pi}{4}, \frac{\pi}{2} - \frac{a_n}{n}\right), \quad J_n^{(1)} = \left(-\frac{\pi}{2} + \frac{b_n}{n}, \frac{\pi}{4}\right);$$

(2,3e) $$\lambda_n = \sqrt{(n(n + \alpha + \beta + 1))};$$

(2,3f) $$\varrho(x) = \lambda_n^2 + \frac{1 - 4\alpha^2}{4x^2}, \quad \varrho_1(x) = \lambda_n^2 + \frac{1 - 4\beta^2}{4x^2};$$

(2,3g) $z_1$ and $z_2$, $z_1 < z_2$ are arbitrary two consecutive zeros of the function

$$Q_n(\sin z).$$

Assertion.

(2,3h) $$[z_1, z_2] \subset J_n \Rightarrow z_2 - z_1 = \pi\varrho^{-1/2}\left(\frac{\pi}{2} - z_1\right) + \delta_1^{(n)}$$

and

(2,3i) $$[z_1, z_2] \subset J_n^{(1)} \Rightarrow z_2 - z_1 = \pi\varrho_1^{-1/2}\left(-\frac{\pi}{2} + z_1\right) + \delta_2^{(n)}.$$

Herein

(2,3j) $$|\delta_1^{(n)}| < cn^{-2}(na_n^{-3} + 1),$$

(2,3k) $$|\delta_2^{(n)}| < cn^{-2}(n|b_n|^{-3} + 1),$$

where $c$ is a constant independent of $n$, $a_n$, $b_n$, $z_1$ and $z_2$, that is, $c$ is the same number for any two consecutive zeros $z_1$, $z_2$ located in $J_n$ and $J_n^{(1)}$ respectively.

For the proof see Chapter 7.

**2,4.** *Let $\delta \in (0, \pi/4)$ be a constant independent of $n$ and*

(2,4a) $$J_\delta = \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right).$$

*Then in terms of the notation of Section 2,3*

(2,4b) $$[z_1, z_2] \subset J_\delta \Rightarrow z_2 - z_1 = \frac{\pi}{n} + \vartheta_n$$

*where*

(2,4c) $$|\vartheta_n| < cn^{-2},$$

*$c$ is a constant with the same properties as that in (2,3j) and (2,3k).*

For the proof see Chapter 7.

**2,5.** *For the zeros of the function* $J_n(\sin z)$ *the following inequalities hold if we employ the notation introduced in Section* 2,3

(2,5a) $$|\delta_1^{(n)}| < cn^{-2}(na_n^{-3} + n^{-1}),$$

(2,5b) $$|\delta_2^{(n)}| < cn^{-2}(n|b_n|^{-3} + n^{-1}),$$

*where* $\delta_1^{(n)}$, $\delta_2^{(n)}$ *are defined by* (2,3h) *and* (2,3i) *respectively.*

For the proof see Chapter 7.

## 3. A TRANSFORMATION OF THE FUNDAMENTAL DIFFERENTIAL EQUATION

**3,1.** We shall employ the following notations

(3,1a) $$z = \arcsin x,$$

(3,1b) $$y' = \frac{dy}{dz}, \quad y'' = \frac{d^2y}{dz^2},$$

(3,1c) $$\omega(z) = (1 + \alpha + \beta)\,\mathrm{tg}\, z + (\alpha - \beta) \sec z,$$

(3,1d) $$J(x) = (1 - x)^\alpha (1 + x)^\beta,$$

(3,1e) $$q(x) = \sqrt{(\cos z\, J(\sin z))} = \exp\left[-\frac{1}{2}\int_0^z \omega(t)\, dt\right],$$

(3,1f) $$\gamma(z) = \tfrac{1}{2}[\omega'(z) - \tfrac{1}{2}\omega^2(z)],$$

(3,1g) $$\alpha_n(z) = \lambda_n^2 + a_n(\sin z) + \gamma(z) - \tfrac{1}{2}[b_n'(\sin z) + u''(\sin z)]\cos^2 z - - \tfrac{1}{4}[b_n(\sin z) + u'(\sin z)]\,\{[b_n(\sin z) + u'(\sin z)]\cos^2 z - 2\omega(z)\cos z - 2\sin z\}.$$

$$\left(\text{Here } b_n'(x) = \frac{db_n(x)}{dx},\ u^{(k)}(\sin z) = \frac{d^k[u(x)]}{dx^2} \quad (k = 1, 2).\right)$$

(3,1h) $$q_n(z) = Q_n(\sin z)\, q(z) \exp\left\{\frac{1}{2}\int_{\pi/2}^z [b_n(\sin t) + u'(\sin t)] \cos t\, dt\right\}.$$

**3,2.** *In the above notation the function* $Q_n(\sin z)$ *is a solution of the differential equation*

(3,2a) $$y'' + \{[u'(\sin z) + b_n(\sin z)]\cos z - \omega(z)\}\, y' + [\lambda_n^2 + a_n(z)]\, y = 0$$

*and the function $q_n(z)$ satisfies the differential equation*

(3,2b)
$$y'' + \alpha_n(z)\, y = 0\,.$$

Proof follows from (1, 3a).

**3,3.** Remark. In the following all the assertions are derived for $x \in [0, 1]$, that is for $z \in [0, \pi/2]$. The same assertions hold for $z \in [-\pi/2, 0]$ if we replace $\alpha$ by $\beta$.

**3,4.** *For* $\zeta \to 0+$

(3,4a)
$$q\left(\frac{\pi}{2} - \zeta\right) = 2^{(\beta-\alpha)/2} \,.\, \zeta^{\alpha+1/2}[1 + O(\zeta^2)]\,,$$

(3,4b)
$$\omega\left(\frac{\pi}{2} - \zeta\right) = (1 + 2\alpha)\,\zeta^{-1} - \tfrac{1}{6}(\alpha + 3\beta + 2)\,\zeta + O(\zeta^3)\,,$$

(3,4c)
$$\gamma\left(\frac{\pi}{2} - \zeta\right) = \tfrac{1}{4}(1 - 4\alpha^2)\,\zeta^{-2} + j + O(\zeta^2)\,,$$

where $j$ is defined by (2,2a).

Proof. Trivial.

**3,5.** *For brevity, put*

(3,5a)
$$\omega_n(\zeta) = \alpha_n\left(\frac{\pi}{2} - \zeta\right) - \lambda_n^2 + \frac{4\alpha^2 - 1}{4\zeta^2}\,.$$

*Then*

(3,5b)
$$\zeta \in \left[0\ \frac{\pi}{2}\right] \Rightarrow |\omega_n(\zeta)| < c_1 n\,.$$

The proof follows from (3,5a), (1,3c), (1,3d), and (1,3e).

**3,6.** *Let* $|\alpha| > \frac{1}{2}$. *Denote by* $\alpha^{(n)}$ *the greatest real zero of the function* $\alpha_n(z)$ *defined by* (3,1g). *Then for* $n \to +\infty$

(3,6a)
$$\alpha^{(n)} = \frac{\pi}{2} - \frac{\alpha_1}{n}\left[1 + O\left(\frac{1}{n}\right)\right],$$

*where for brevity*

(3,6b)
$$\alpha_1 = \tfrac{1}{2}\sqrt{(4\alpha^2 - 1)}\,.$$

Remark. For almost all values of $n$ there exists one and only one positive zero of $\alpha_n(z)$ (provided $|\alpha| > \frac{1}{2}$).

Proof. According to (3,5a) and (3,5b) it is

$$\frac{\pi}{2} - \alpha^{(n)} = \frac{\lambda_n^{-1}}{2}\left\{(4\alpha^2 - 1)\Big/\left[1 + \lambda_n^{-2}\omega_n\left(\frac{\pi}{2} - \alpha^{(n)}\right)\right]\right\}^{1/2} = \frac{\alpha_1}{n}\left[1 + O(n^{-1})\right].$$

**3,7.** *Let $|\alpha| > \frac{1}{2}$ and let $\alpha_0 > \alpha_1$ be a constant independent of $n$, where $\alpha_1$ is defined by (3,6b). Then for $z \in [0, \pi/2 - \alpha_0/n]$*

(3,7a) $$0 < \alpha_n^{-1}(z) < c_1 n^{-2}$$

*for almost all values of n.*

*If $\alpha \leqq -\frac{1}{2}$, then (3,7a) holds for every $\alpha_0 > 0$.*

Proof. Put

$$f(x) = \frac{1 - 4\alpha^2}{4x^2}.$$

Hence $f(\alpha_1) = -1$. Since $f(x)$ is an increasing function for $x > 0$, there exists in virtue of (3,5a) and (3,5b) a constant $c > 0$ independent of $n$ such that for almost all values of $n$

$$\zeta \in \left(\frac{\alpha_0}{n}, \frac{\pi}{2}\right) \Rightarrow \alpha_n\left(\frac{\pi}{2} - \zeta\right) = \lambda_n^2 + f(\zeta) + \omega_n(\zeta) > \lambda_n^2 + n^2[f(\alpha_0) - f(\alpha_1)] +$$

$$+ n^2 f(\alpha_1) - cn = \lambda_n^2 - n^2 - cn + \frac{\alpha_0^2 - \alpha_1^2}{4\alpha_0^2\alpha_1^2}(4\alpha^2 - 1)\, n^2 > \frac{\alpha_0^2 - \alpha_1^2}{8\alpha_1^2\alpha_0^2}(4\alpha^2 - 1)\, n^2.$$

**3,8.** *For brevity, put*

(3,8a) $$\psi_n(x) = q_n\left(\frac{\pi}{2} - x\right).$$

*Then for $x \to 0+$*

(3,8b) $$\psi_n(x) = 2^{(\beta-\alpha)/2} x^{\alpha+1/2}\, Q_n(1)\left[1 + O(x^2)\right],$$

*where*

(3,8c) $$Q_n(1) > 0\,.$$

Proof. For brevity, put

(1) $$\varepsilon_n(x) = \exp\left\{-\frac{1}{2}\int_{\pi/2-x}^{\pi/2}[b_n(\sin t) + u'(\sin t)\cos t\, dt\right\} =$$

$$= \exp\left\{-\frac{1}{2}\int_0^x [b_n(\cos t) + u'(\cos t)]\sin t\, dt\right\} = 1 + O(x^2) \quad \text{for} \quad x \to 0+\,.$$

Further

$$(2) \qquad Q_n(\cos x) = Q_n(1) + O(x^2).$$

Since

$$\psi_n(x) = Q_n(\cos x)\, q\left(\frac{\pi}{2} - x\right) \varepsilon_n(x),$$

(3.8b) follows from (1), (2) and (3,4a).

By a well known theorem

$$Q_n(x) \neq 0 \quad \text{for} \quad x \geqq 1$$

and in virtue of (1,1a) it is $Q_n(+\infty) = +\infty$. This shows that (3,8c) is true.

## 4. LEMMAS

**4,1.** In the following we employ the Bessel functions $I_\alpha(x)$ of the order $\alpha$ and of the first kind as well as the Bessel functions $Y_\alpha(x)$ of the order $\alpha$ and the second kind.

It is well known that

$$(4,1a) \qquad I_\alpha(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{\alpha+2\nu}}{\nu!\,\Gamma(\alpha + \nu + 1)}$$

and provided $\alpha \geqq 0$ is an integer,

$$(4,1b) \qquad Y_\alpha(x) = \frac{2}{\pi}\left[C + \lg\frac{x}{2}\right] I_\alpha(x) - \frac{1}{\pi}\sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{x}{2}\right)^{\alpha+2\nu}}{\nu!\,(\nu + \alpha)!}\,\sigma_\nu - S_\alpha(x).$$

Herein $C$ is the Euler constant and

$$\alpha > 0 \Rightarrow \sigma_0 = \sum_{k=1}^{\alpha} \frac{1}{k}, \quad \alpha = 0 \Rightarrow \sigma_0 = 1,$$

$$\nu > 0 \Rightarrow \sigma_\nu = \sum_{k=1}^{\nu} \frac{1}{k} + \sum_{k=1}^{\nu+\alpha} \frac{1}{k},$$

$$S_0(x) = 0, \quad \alpha > 0 \Rightarrow S_\alpha(x) = \frac{1}{\pi}\sum_{\nu=0}^{\alpha-1} \frac{(\alpha - \nu - 1)!\left(\frac{x}{2}\right)^{2\nu-\alpha}}{\nu!}.$$

**4,2.** Put

$$(4,2a) \qquad v(x) = \sqrt{(x)}\, I_\alpha(x)$$

and if $\alpha$ is not an integer,

(4,2b) $$w(x) = \sqrt{(x)}\, I_{-\alpha}(x)\,.$$

If $\alpha$ is an integer, then

(4,2c) $$w(x) = \sqrt{(x)}\, Y_\alpha(x)\,.$$

$v(x)$ and $w(x)$ are linearly independent solutions of the differential equation

(4,2d) $$y'' + \left(1 + \frac{1 - 4\alpha^2}{4x^2}\right) y = 0\,.$$

(See [I] pp. 29–30.)

It is easily seen that for any real number $k$ the functions $v(kx)$ and $w(kx)$ are linearly independent solutions of the differential equation

(4,2e) $$y'' + \left[k^2 + \frac{1 - 4\alpha^2}{4x^2}\right] y = 0\,.$$

(See [I] p. 31.)

**4,3.** The following theorem will be used:

*Let $p(x)$ and $q(x) < 0$ be real functions continuous on the interval $(a, b)$ and let $\varphi(x)$ be a solution of the differential equation*

(4,3a) $$y'' + p(x)\, y' + q(x)\, y = 0\,.$$

*Then the function $\varphi(x)\,.\,\varphi'(x)$ has at most one zero in the closed interval $[a, b]$. Herein $a$ or $b$ are also zeros of $\varphi(x)\,\varphi'(x)$ if for $i = 0,1$*

$$\lim_{x \to a+} \varphi^{(i)}(x) = 0 \quad \text{or} \quad \lim_{x \to b-} \varphi^{(i)}(x) = 0\,.$$

Proof. (See [2] pp. 164–165.)

**4,4.** *Let $\{x_{\alpha,n}\}_{n=1}^{\infty}$ and $\{x'_{\alpha,n}\}_{n=1}^{\infty}$ be the increasing sequences of all the positive zeros of the functions $v(x)$ and $v'(x)$ respectively.*

*Let $\{\zeta_n\}_{n=0}^{\infty}$ and $\{\zeta'_n\}_{n=0}^{\infty}$ be the increasing sequences of all the positive zeros of the functions $\psi_n(x)$ and $\psi'_n(x)$ respectively.*[2]

*If $|\alpha| > \frac{1}{2}$, then*

(4,4a) $$x_{\alpha,1} > x'_{\alpha,1} > \tfrac{1}{2}\sqrt{(4\alpha^2 - 1)} = \alpha_1\,.^{3)}$$

[2]) See (3,8a).
[3]) See (3,6b).

*and*

$$(4,4b) \qquad \tilde{\alpha}_n\left(\frac{\pi}{2} - \zeta_1\right) > \alpha_n\left(\frac{\pi}{2} - \zeta_1'\right) > 0\,.$$

Proof. Since

$$v(0) = \psi_n(0) = 0$$

and $y = v(x)$ is a solution of the equation (4,2d) our assertion is a consequence of theorem in Section 4,3.

**4,5.** *Let* $v(x)$ *and* $w(x)$ *be the functions defined by* (4,2a), (4,2b) *and* (4,2c) *respectively and let* $\psi_n(x)$ *be defined by* (3,8b).

*For brevity, put*

$$(4,5a) \qquad W(x, t) = v(x)\, w(t) - v(t)\, w(x)\,,$$

$$(4,5b) \qquad l^{-1} = v'(x)\, w(x) - v(x)\, w'(x)\,,$$

$$(4,5c) \qquad l_n = \sqrt{(\lambda_n^2 + \tau_n)}\,,$$

*where* $\lambda_n = \sqrt{(n(n + \alpha + \beta + 1))}$ *and*

$$(4,5d) \qquad \tau_n = O(n)$$

*is a real number depending on n.*

*Further, put*

$$(4,5e) \qquad \psi_n = \lim_{x\to 0+} \frac{v(l_n x)}{\psi_n(x)} = \frac{2^{-(\alpha+\beta)/2} l_n^{\alpha+1/2}}{\Gamma(\alpha + 1)\, Q_n(1)}\,,$$

$$(4,5f) \qquad \chi_n(x) = \psi_n\, \psi_n(x)$$

and

$$(4,5g) \qquad \beta_n(t) = \omega_n(t) - \tau_n$$

*where* $\omega_n(t)$ *is defined by* (3,5a).

*Then for* $x \in (0, 1)$

$$(4,5h) \qquad \chi_n(x) = v(l_n x) - \varrho_n(x)$$

*where*

$$(4,5i) \qquad \varrho_n(x) = l l_n^{-1} \int_0^x \beta_n(t)\, W(l_n x, l_n t)\, \chi_n(t)\, dt\,.$$

Proof. 1. Denote by $k_i$ $(i = 1, 2, \ldots)$ positive constants independent of $x$ and $t$ in the interval $[0, 1]$. (They may depend on $n$.)

In virtue of (3,8b) and (3,5b) we may write for $t \in (0, 1)$

(1) $$|\chi_n(t)| < k_1 t^{\alpha+1/2}, \quad |\beta_n(t)| < k_2 .$$

By applying (4,1a) and (4,1b) we deduce that for $x \in (0, 1)$ and $t \in [0, 1)$ and $x > t$

(2) $$|W(l_n x, l_n t)| < k_3\, \delta(x, t) = k_3[(xt^{-1})^\alpha + (x^{-1}t)^\alpha] \sqrt{(xt)}\, \lg^{m_0} \left|\frac{ex}{t}\right| ,$$

where $m_0 = 1$ if $\alpha = 0$, and $m_0 = 0$ if $\alpha \neq 0$.

From (1) and (2) it follows for $x \in (0, 1)$

(3) $$|\varrho_n(x)| < k_4 \int_0^x t^{\alpha+1/2}\, \delta(x, t)\, \mathrm{d}t < k_5 x^{\alpha+5/2} .$$

2. The function $\chi_n(x)$ defined by (4,5f) is a solution of the differential equation

$$y'' + \alpha_n \left(\frac{\pi}{2} - x\right) y = 0 .$$

Hence

(4) $$\chi_n''(x) + \left[l_n^2 + \frac{1 - 4\alpha^2}{4x^2}\right] \chi_n(x) = -\beta_n(x)\, \chi_n(x) .$$

By (4) we derive the equation

(5) $$\chi_n(x) = C_1\, v(l_n x) + C_2\, w(l_n x) - \varrho_n(x) ,$$

where $C_1$ and $C_2$ are constants.

Let $\alpha$ be non integer. Making use of (3,8b), (4,1a), (4,5e) and (3) we deduce by (5) that for $x \to 0+$

$$\frac{(l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha + 1)} + O(x^{\alpha+5/2}) = \frac{C_1 (l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha + 1)} + O(x^{\alpha+5/2}) + \frac{C_2 (l_n x)^{-\alpha+1/2}}{2^{-\alpha} \Gamma(1 - \alpha)} [1 + O(x^2)] .$$

Hence

(6) $$C_1 + \frac{2^{2\alpha} \Gamma(\alpha + 1)}{\Gamma(1 - \alpha)}\, l_n^{-2\alpha} x^{-2\alpha} [1 + O(x^2)]\, C_2 = 1 + O(x^2) .$$

From (6) it is easily seen that

(7) $$\alpha > 0 \Rightarrow C_2 = O(x^{2\alpha}) \Rightarrow C_2 = 0 , \quad C_1 = 1$$

and

$$\alpha < 0 \Rightarrow C_1 = 1 + O(x^{-2\alpha}) \Rightarrow C_1 = 1 , \quad C_2 = O(x^{2-2\alpha}) \Rightarrow C_2 = 0 .$$

If $\alpha$ is an integer, then by (3,8b), (4,1b) and (3) we deduce that for $x \to 0+$

$$\frac{(l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha+1)} + O(x^{\alpha+5/2}) = \frac{C_1 (l_n x)^{\alpha+1/2}}{2^\alpha \Gamma(\alpha+1)} + O(x^{\alpha+5/2}) +$$

$$+ \frac{1}{\pi}\left[\frac{(l_n x)^{\alpha+1/2}}{2^{\alpha-1}\Gamma(\alpha+1)} \lg x + 2^\alpha(\alpha-1)!\,(l_n x)^{-\alpha+1/2}\right][1 + O(x^2)]\, C_2 .$$

Hence we deduce $C_1 = 1$, $C_2 = 0$ by a similar argument as above.

**4,6.** *Let $a > 0$ be an arbitrary number independent of $n$ and*

(4,6a) $$I_a = \left(0, \frac{a}{n}\right).$$

*Further denote by $\gamma_n(x)$ a real function defined in the interval $I_a$ such that*

(4,6b) $$t \in I_a \Rightarrow |\gamma_n(t)| < \gamma_n .$$

*Put*

(4,6c) $$\sigma_n(x) = \int_0^x \gamma_n(t)\, W(l_n x, l_n t)\, \chi_n(t)\, \mathrm{d}t ,$$

*where $\chi_n(x)$ is defined by* (4,5f).

*Then*

(4,6d) $$x \in I_a \Rightarrow |\sigma_n(x)| < c_1 n^{-1} \gamma_n .$$

*From* (4,5i) *and* (4,6d) *we deduce that*

(4,6e) $$x \in I_a \Rightarrow |\varrho_n(x)| < c_2 n^{-1} .$$

Proof. 1. For brevity, put

(1) $$l_n(x) = x^{-\alpha-1/2}\, \chi_n(x), \quad S_n = \sup_{x \in I_a} |l_n(x)| .$$

Making use of (4,6b) and (2) in Section 4,5, we obtain from (4,6c)

(2) $$x \in I_a \Rightarrow |\sigma_n(x)| < c_3 \gamma_n x S_n x^{\alpha+1/2} < c_4 \gamma_n n^{-1} S_n x^{\alpha+1/2} .$$

2. Put $\gamma_n(t) = \beta_n(t)$, where $\beta_n(t)$ is defined by (4,5g). In this case we may put $\gamma_n = c_5 n$ so that we obtain from (4,5i) and (2)

(3) $$x \in I_a \Rightarrow |\varrho_n(x)| < c_6 n^{-1} n n^{-1} x^{\alpha+1/2} S_n < c_7 n^{-1} x^{\alpha+1/2} S_n .$$

Since

(4) $$x \in I_a \Rightarrow |v(l_n x)| < c_8 (l_n x)^{\alpha+1/2}$$

and by (4,5h)

$$l_n(x) = [v(l_n x) - \varrho_n(x)]\, x^{-\alpha-1/2} \tag{5}$$

we deduce by (2) and (5)

$$S_n < c_9 n^{\alpha+1/2} + c_{10} n^{-1} S_n \Rightarrow S_n < c_{11} n^{\alpha+1/2}\,.$$

Applying this result we obtain (4,6d) from (2) and (4,6e) from (3).

**4,7.** *Let $v(x)$ be defined by* (4,2a) *and let $x_{\alpha,k}$ $(k = 1, 2, \ldots)$ be the zeros of $v(x)$ introduced in Section* 4,4. *Let $A_n > 0$ satisfy the condition*

(4,7a) $$A_n = o(1) \quad for \quad n \to +\infty\,.$$

*If*

(4,7b) $x_{\alpha,0} = 0,\ x_{\alpha,k} + \eta n \in (x_{\alpha,k-1} + c_1,\ x_{\alpha,k+1} - c_1)$ *and* $|v(x_{\alpha,k} + n\eta)| < A_n\,,$

*then*

(4,7c) $$|\eta| < c_2 n^{-1} A_n\,.$$

Proof. For brevity, put $x_{\alpha,k} = x_k$ and $x_k + n\eta = b$.

Let $I_\eta$ be the interval $(b, x_k)$ if $\eta < 0$ or $(x_k, b)$ if $\eta > 0$. By (4,7a) and (4,7b) we deduce

$$x \in I_\eta \Rightarrow |v(x)| < A_n\,. \tag{1}$$

Further

$$v(b) = n\eta\, v'(x_k) + \tfrac{1}{2} n^2 \eta^2\, v''(\xi)\,, \tag{2}$$

where

$$\xi \in I_\eta\,. \tag{3}$$

From the equation (4,2d) we obtain

$$v''(\xi) = \left[\frac{4\alpha^2 - 1}{4\xi^2} - 1\right] v(\xi)\,. \tag{4}$$

Making use of (4), and (1) we deduce

$$|v''(\xi)| < c_3 |v(\xi)| < c_4 A_n\,. \tag{5}$$

Since $v'(x_k) \neq 0$ it follows from (2), (5) and (4,7a) that

$$A_n > |v(b)| > n|\eta|\, |v'(x_k)| - c_5 |v''(\xi)| > n\eta |v'(x_k)| - c_6 A_n$$

for almost all values of $n$.

**4,8.** *Following the notation of Section* 4,6 *we put*

$$h_n(x) = v(l_n x) + \eta_n(x)\,, \tag{4,8a}$$

*where* $l_n$ *is defined by* (4,5c)

$$x \in I_a \Rightarrow |\eta_n(x)| < A_n\,. \tag{4,8b}$$

*Here* $A_n$ *satisfies* (4,7a).

*Let* $\{\xi_n\}_{n=1}^{N}$ *be the increasing sequence of all the zeros of the function* $h_n(x)$ *contained in the interval* $I_a$. *Then the following assertions are true*:

a) *For every positive integer* $k$ *there exists an integer* $r > 0$ *such that for* $n \to +\infty$

$$\xi_k = \frac{x_{\alpha,r}}{l_n}\,[1 + O(A_n)]\,. \tag{4,8c}$$

b) *For every integer* $m > 0$ *there exists an integer* $s$ *such that for* $n \to +\infty$

$$\xi_s = \frac{x_{\alpha,m}}{l_n}\,[1 + O(A_n)]\,. \tag{4,8d}$$

Proof. 1. Let $\{x'_{\alpha,n}\}_{n=1}^{\infty}$ be the increasing sequence of all the positive zeros of the function $v'(x)$.

From (4,8b) and (4,7a) we deduce the following assertion A: For every integer $\nu > 0$ there is at least one zero of the function $h_n(x)$ in the interval $(x'_{\alpha,\nu}/l_n, x_{\alpha,\nu+1}/l_n)$.

2. Put

$$\xi_k = \frac{x_{\alpha,r}}{l_n} + l_n^{-1} n\eta\,, \tag{1}$$

where $x_{\alpha,r}$ is the zero of the function $v(l_n x)$ nearest to the number $\xi_k$. From the above proposition

$$\xi_k < \frac{x'_{\alpha,k+1}}{l_n} < \frac{x_{\alpha,k+2}}{l_n} \in I_a\,. \tag{2}$$

From (2) it is obvious that $r \leqq k + 2$.

If $a > x_{\alpha,k+2}$ it follows from (4,8b) that

$$|\eta_n(\xi_k)| < A_n\,. \tag{3}$$

By (4,8a) and (1) we deduce that

$$0 = h_n(\xi_k) = v(x_r + n\eta) + \eta_n(\xi_k)\,. \tag{4}$$

Hence we obtain as a consequence of (3) and (4,8b) that

$$|v(x_r + n\eta)| < A_n\,. \tag{5}$$

The proposition of Section 4,7 yields

$$|\eta| < A_n n^{-1} .$$

This inequality shows that (4,8c) is true.

3. Let

$$\frac{x_{\alpha,m}}{l_n} = \xi_s - n l_n^{-1} \eta' , \tag{6}$$

where $\xi_s$ is a zero of the function $h_n(x)$ nearest to the number $x_{\alpha,m}/l_n$. From the above assertion A we see that

$$a > x'_{\alpha,m+2} \Rightarrow \xi_s < \frac{x'_{\alpha,m+2}}{l_n} \in I_a . \tag{7}$$

Making use of (4,8a) we obtain

$$0 = h_n(\xi_s) = v(x_{\alpha,m} + n\eta') + \eta_n(\xi_s) .$$

Hence, in virtue of (7) and (4,8b)

$$|v(x_m + n\eta')| < A_n \tag{8}$$

Hence by the statement of Section 4,7

$$|\eta'| < n^{-1} A_n . \tag{9}$$

(7) and (9) establish (4,8d).

## 5. PROOF OF (2,1a) AND (2,1b)

**5,1.** *In the notation introduced in Section 4,4, for $k = 1, 2, \ldots$ independent of $n$ it is*

$$\zeta_k = \frac{x_{\alpha,k}}{n} [1 + O(n^{-1})] \quad for \quad n \to +\infty . \tag{5,1a}$$

Proof. 1. The zeros of the function $\psi_n(x)$ coincide with the zeros of the function $\chi_n(x)$ defined by (4,5f). Let $I_a$ be defined by (4,6a) and choose $a$ sufficiently large.

In virtue of (4,5h) and (4,6e) the theorem of Section 4,8 yields for $k = 1, 2, \ldots$ and $m = 1, 2, \ldots$ provided that $\zeta_k \in I_a$ and $x_{\alpha,m}/n \in I_a$,

$$\zeta_k = \frac{x_{\alpha,r}}{n} [1 + O(n^{-1})] \tag{1}$$

and

$$\zeta_s = \frac{x_{\alpha,m}}{n} [1 + O(n^{-1})] . \tag{2}$$

Herein $x_{\alpha,r}/l_n$ is the zero of $v(l_n x)$ nearest to the number $\zeta_k$ and $\zeta_s$ is the zero of $\chi_n(x)$ nearest to the number $x_{\alpha,m}/l_n$.

2. Put in (1) $k = 1$ and in (2) $m = 1$. Then

$$n\zeta_1 \geqq x_{\alpha,1} + O(n^{-1}) \tag{3}$$

and

$$n\zeta_1 \leqq x_{\alpha,1} + O(n^{-1}) . \tag{4}$$

From (3) and (4) we see that

$$\zeta_1 = \frac{x_{\alpha,1}}{n} [1 + O(n^{-1})] . \tag{5}$$

Hereby (5,1a) is established for $k = 1$.

3. Let $\omega_n(x)$ be defined by (3,5a) and put

$$s_n = \sup_{x \in [0,\pi/2]} |\omega_n(x)| . \tag{6}$$

In virtue of (3,5b) we may choose $k_1 > 1$ independent of $n$ and $\sigma_n$ such that

$$k_1 n > \sigma_n > s_n . \tag{7}$$

Put

$$\lambda = \sqrt{(\lambda_n^2 - \sigma_n)} . \tag{8}$$

(5) enables us to choose $\sigma_n$ so that

$$\frac{x_{\alpha,1}}{\lambda} > \zeta_1 . \tag{9}$$

Since the functions $v(\lambda x)$ and $\chi_n(x)$ are solutions of the differential equations

$$y'' + \left[\lambda^2 + \frac{1 - 4\alpha^2}{4x^2}\right] y = 0 \tag{10}$$

and

$$y'' + \left[\lambda_n^2 + \frac{1 - 4\alpha^2}{4x^2} + \omega_n(x)\right] y = 0 \tag{11}$$

respectively it follows by the well-known Sturm's comparison theorem in virtue of (9) that in the interval $[0, \zeta_k]$ there are at most $(k - 1)$ zeros of the function $v(\lambda x)$. Hence we obtain for the number $k$ and $r$ in (1)

$$r \leqq k . \tag{12}$$

3. Further, put

(13) $$k_2 n > \mu_n > s_n\,, \quad \mu = \sqrt{(\lambda_n^2 + \mu_n)}\,,$$

where $k_2$ does not depend on $n$ and $s_n$ is defined by (6). Choose $\mu_n$ so that

(14) $$\zeta_1 > \frac{x_{\alpha,1}}{\mu}\,.$$

Then there are at least $(k - 1)$ zeros of $v(\mu x)$ in the interval $[0, \zeta_k]$. Hence by (1)

(15) $$\zeta_k = \frac{x_{\alpha,t}}{n}\,[1 + O(n^{-1})]\,,$$

where

(16) $$t \geqq k\,.$$

From (1) and (15) we deduce that

$$0 = x_{\alpha,r} - x_{\alpha,t} + O(n^{-1})\,.$$

Hence

(17) $$x_{\alpha,r} = x_{\alpha,t} \Rightarrow r = t\,.$$

(12), (16) and (17) show that $r = k$.

**5,2.** The proof of (2,1b). By (5,1a) we deduce

$$x_{n-k}^{(n)} = \sin\left(\frac{\pi}{2} - \zeta_{k+1}\right) = 1 - \frac{x_{\alpha,k+1}}{2n^2}\,[1 + O(n^{-1})]$$

for $n \to +\infty$.

**5,3.** For the proof of (2,1a) see Remark 3,3.

## 6. PROOF OF (2,2b) AND (2,2c)

**6,1.** 1. Put $Q_n(x) = J_n(x)$. Then by (3,5a)

(1) $$\omega_n(t) = \gamma(t) - \frac{1 - 4\alpha^2}{4t^2}\,.$$

Put in (4,5c) and (4,5g)

(2) $$l_n = (\lambda_n^2 + j)^{1/2}\,,$$

(3) $$\beta_n(t) = \omega_n(t) - j\,.$$

where $j$ is defined by (2,1a).

Let $I_a$ be defined by (4,6a) and $a$ sufficiently large. It is easly to see from (3,4c) and (1) that

(4) $$t \in I_a \Rightarrow |\beta_n(t)| < c_1 n^{-2}.$$

Then by (4,5i) and (4,6d)

(5) $$x \in I_a \Rightarrow |\varrho_n(x)| < c_2 n^{-4}$$

for in this case $\gamma_n = c_3 n^{-2}$.

Denote by $\{\zeta_k\}_{k=1}^n$ the increasing sequence of all the zeros of $J_n(\sin z)$.

By the theorem of Section 4,8 and by (5) we deduce that for every $k = 1, 2, \ldots$ there exists an ínteger $r > 0$ such that

(6) $$\zeta_k = \frac{x_{\alpha,r}}{l_n} + O(n^{-5}).$$

By (5,1a) we have

(7) $$\zeta_k = \frac{x_{\alpha,k}}{n} + O(n^{-2}).$$

From (6) and (7) it follows that

$$0 = x_{\alpha,r} - x_{\alpha,k} + O(n^{-1}).$$

Hence

$$x_{\alpha,r} = x_{\alpha,k} \Rightarrow r = k$$

so that by (6)

(8) $$\zeta_k = \frac{x_{\alpha,k}}{l_n} + O(n^{-5}).$$

2. Let $Q_n(x) = J_n(x)$. Then

(9) $$x_{n-k+1}^{(n)} = \cos \zeta_k = 1 - \frac{\zeta_k^2}{2} + \frac{\zeta_k^4}{24} + O(n^{-6}).$$

From (2) it is obvious that

(10)
$$n^2 l_n^{-2} = 1 - \frac{\alpha + \beta + 1}{n} - \frac{j}{n^2} - \left[\frac{\alpha + \beta + 1}{n} + \frac{j}{n^2}\right]^2 - \frac{(\alpha + \beta + 1)^3}{n^3} + O(n^{-4}) =$$
$$= 1 - \frac{\alpha + \beta + 1}{n} - \frac{(\alpha + \beta + 1)^2 + j}{n^2} - \frac{(\alpha + \beta + 1)\,[2j + (\alpha + \beta + 1)^2]}{n^3} +$$
$$+ O(n^{-4}).$$

Further

$$(11) \qquad n^4 l_n^{-4} = 1 - \frac{2(\alpha + \beta + 1)}{n} + O(n^{-2}).$$

From (8)–(11) we may deduce (2,2c).
As for (2,2b), see Remark 3,3.

## 7. PROOF OF THE INEQUALITIES IN SECTIONS 2,3; 2,4 AND 2,5

**7,1.** *In the notations introduced in Section* 2,3

$$(7{,}1a) \qquad z \in J_n \Rightarrow c_1 n^2 < \alpha_n(z) < c_2 n^2 .$$

Proof. (7,1a) is a consequence of (3,5a), (3,5b). See also (3,7a).

**7,2.** *Let $z_1$ and $z_2$ be defined by* (2,3g). *Then*

$$(7{,}2a) \qquad (z_1, z_2) \subset J_n \Rightarrow z_2 - z_1 < c_1 n^{-1} .$$

Proof. Employing Sturm's comparison theorem we obtain from the differential euqation $y'' + \alpha_n(z)\, y = 0$

$$(1) \qquad z_2 - z_1 < \pi \sup_{z \in J_n} \alpha_n^{-1/2}(z) .$$

Now, (7,2a) is a consequence of (1) and (7,1a).

**7,3.** *In the notation of Section* 2,3

$$(7{,}3a) \qquad [z_1', z_2'] \subset [z_1, z_2] \Rightarrow \left|\varrho\left(\frac{\pi}{2} - z_1'\right) - \varrho\left(\frac{\pi}{2} - z_2'\right)\right| < c_1 n^2 a_n^{-3} .$$

*Here $c_1$ does not depend on $z_i$, $z_i'$ $(i = 1, 2)$.*

Proof. For brevity, put

$$\xi_i' = \frac{\pi}{2} - z_i' \quad (i = 1, 2) .$$

From (2,3d) it follows

$$\xi_i' > \frac{a_n}{n} .$$

Now, (7,2a) yields

$$|\varrho(\xi_1') - \varrho(\xi_2')| = |\alpha^2 - \tfrac{1}{4}| \frac{(\xi_1' - \xi_2')(\xi_1' + \xi_2')}{\xi_1'^2 \cdot \xi_2'^2} < c_2 n^{-1} \xi_2'^{-3} < c_3 n^2 a_n^{-3} .$$

7,4 *According to the notation introduced in the preceding chapter*

(7,4a) $$\delta_n = |\alpha_n^{-1/2}(z_1') - \alpha_n^{-1/2}(z_2')| < c_1 n^{-2}(n a_n^{-3} + 1).$$

Proof. Making use of (7,3a), (3,5a) and (3,5b), we obtain

$$|\alpha_n(z_1') - \alpha_n(z_2')| = |\varrho(\xi_2') - \varrho(\xi_1') + \omega_n(\xi_2') - \omega_n(\xi_1')| < c_2 n(n a_n^{-3} + 1).$$

Further, it follows from (7,1a) and (7,2a) that

$$\delta_n = |\alpha_n(\xi_1') - \alpha_n(\xi_2')| [\alpha_n(\xi_1') \alpha_n(\xi_2')]^{-1/2} [\sqrt{\alpha_n(\xi_1')} + \sqrt{\alpha_n(\xi_2')}]^{-1} <$$
$$< c_3 n^{-2}(n a_n^{-3} + 1).$$

7,5. The proof of (2,3i).

Put

$$s_1 = \sup_{z \in (z_1, z_2)} \alpha_n^{-1/2}(z), \quad s_2 = \inf_{z \in (z_1, z_2)} \alpha_n^{-1/2}(z).$$

Making use of Sturm's comparison theorem, we deduce by the differential equation (3,2b)

$$\pi s_2 < z_2 - z_1 < \pi s_1.$$

Hence

(1) $$z_2 - z_1 = \pi s_2 + \vartheta(s_1 - s_2)$$

where $\vartheta \in (0, 1)$. Put

(7,5a) $$s_1 = \alpha_n^{-1/2}(z_1) + \vartheta_1^{(n)}, \quad s_2 = \alpha_n^{-1/2}(z_1) + \vartheta_2^{(n)}, \quad s_1 - s_2 = \vartheta_3^{(n)}.$$

From (7,4a) it follows for $i = 1, 2, 3$

(2) $$|\vartheta_i^{(n)}| < c_1 n^{-2}(n a_n^{-3} + 1).$$

By (7,5a), (7,1a), (3,5a), (3,5b), (1) and (2) we deduce that

(7,5b) $$z_2 - z_1 = \pi \alpha_n^{-1/2}(z_1) + \vartheta_4^{(n)} = \pi \varrho^{-\frac{1}{2}} \left( \frac{\pi}{2} - k_1 \right) + O(n^{-2}) + \vartheta_4^{(n)}$$

where $\vartheta_4^{(n)}$ satisfies (2) for $i = 4$.

7.6 The proof of (2.5a). It follows from (3,5a) for the polynomials $J_n(x)$ that

(1) $$\omega_n(\zeta) = \gamma \left( \frac{\pi}{2} - \zeta \right) + \frac{4\alpha^2 - 1}{4\zeta^2}.$$

Hence

(2) $$\frac{\pi}{2} - \zeta \in J_n \Rightarrow |\omega_n(\zeta)| < c_1.$$

From (2) we deduce by a similar argument as in Section 7,4 that in this case

(3) $$\delta_n < c_2 n^{-1}(a_n^{-3} + n^{-1}),$$

where $\delta_n$ is defined by (7,4a).

By (2) we deduce

(4) $$|\vartheta_i^{(n)}| < c_3 n^{-1} a_n^{-3} \quad (i = 1, 2, 3, 4),$$

where $\vartheta_i^{(n)}$ is defined by equations (7,5a) and (7,5b). (2,5a) is a consequence of (7,5b) and (2).

**7,7.** The proof of (2,4b).

(2,4b) is a consequence of (2,3h) and (2,3i) for

$$(z_1, z_2) \subset \left(-\frac{\pi}{4}, \frac{\pi}{2} - \delta\right) \Rightarrow \alpha_n^{-1/2}(z_1) = \frac{1}{n} + O(n^{-2})$$

and

$$\delta = \frac{a_n}{n} \Rightarrow a_n = \delta n \Rightarrow a_n^{-3} = \delta^{-3} n^{-3}.$$

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