Roy O. Davies
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A BAIRE FUNCTION NOT COUNTABLY DECOMPOSABLE
INTO CONTINUOUS FUNCTIONS

ROY O. DAVIES, Leicester

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In connection with a problem of KARTÁK [1], VRKOČ recently constructed [2] a measurable real function \( f \) on \( I = [0, 1] \) such that \( I \) cannot be partitioned into countably many sets \( A_n \) with each restriction \( f \mid A_n \) continuous. He asked whether for every Baire function there does exist such a partition of \( I \) into Borel sets. Here it will be shown that, on the contrary, there exists a function of Baire class 1 for which there exists no such partition whatever, even into non-Borel sets.

**Theorem 1.** If \( f: A \to I \) is continuous, where \( A \) is a subset of \( I \), then given \( \varepsilon > 0 \) there exists a closed set \( F \subseteq I \times I \) such that \( F \cap \text{Gr}(f) = \emptyset \) and \( m(F_x) \leq 1 - \varepsilon \) for all \( x \in I \).

**Proof.** For each element \( u \in A \), let \( J_u = \{(u, y) : |y - f(u)| < \frac{\varepsilon}{4}\} \) and \( K_u = (\{u\} \times I) \setminus J_u \). Denote by \( E \) the closure of the set \( D = \bigcup\{K_u : u \in A\} \). First, we observe that \( E \cap \text{Gr}(f) = \emptyset \). Indeed, given any point \( (u, f(u)) \in \text{Gr}(f) \), we can choose \( \delta > 0 \) so small that

\[
\forall v \in A \& |v - u| < \delta \Rightarrow |f(v) - f(u)| < \varepsilon/4;
\]

then the open rectangle with centre at \((u, f(u))\), width \( 2\delta \), and height \( \frac{\varepsilon}{4} \) contains no point of \( D \).

Next, we prove that for every \( x \in \overline{A} \), the set \( I \setminus E_x \) is an interval of length at most \( \varepsilon \), open relative to \( I \). Since \( E_x \) is closed, it is enough to show that if \( y_1, y_2 \in I \setminus E_x \) and \( y_1 < y < y_2 \) then (i) \( y_2 - y_1 < \varepsilon \) and (ii) \( y \in I \setminus E_x \). Consider any \( u \in A \) with \( |u - x| < \delta \), where \( \delta \) is the smaller of the distances of \((x, y_1)\) and \((x, y_2)\) from \( E \); then \((u, y_1) \in J_u \) and \((u, y_2) \in J_u \); and (i) follows. Moreover \( |y - f(u)| < \frac{\varepsilon}{4} - \min(y_2 - y, y - y_1) \); hence the open rectangle with centre \((u, y)\), width \( 2\delta \), and height \( 2 \min(y_2 - y, y - y_1) \) contains no point of \( D \), and this establishes (ii).

To construct \( F \) we adjoin to \( E \) a large part of each strip \( S = (c, d) \times I \), where \((c, d)\) is an interval of \( I \setminus \overline{A} \); namely, the whole of \( S \) except for an open “corridor”
(with rectilinear edges) joining the open vertical intervals $G = \{c\} \times (I \setminus E_c)$ and $H = \{d\} \times (I \setminus E_d)$. (If $G = H = \emptyset$ we include the whole of $S$ in $F$; while if, for example, $H = \emptyset$ but $G \neq \emptyset$ then we include the whole of $S$ except for the open triangle joining $G$ to the point $(d, z)$, where $(c, z)$ is the midpoint of $G$.) It is easy to verify that the resulting set $F$ is closed, and it clearly has the other required properties.

**Theorem 2.** Let $f : I \to I$ be such that there is a partition $I = \bigcup_{n=1}^{\infty} A_n$ with each restriction $f \mid A_n$ continuous. Then given $\varepsilon > 0$ there exists a closed set $F \subseteq I \times I$ such that $F \cap \text{Gr}(f) = \emptyset$ and $m(F_x) \geq 1 - \varepsilon$ for all $x \in I$.

**Proof.** Let $\sum \varepsilon_n$ be a convergent series of positive terms with sum less than $\varepsilon$. By Theorem 1 there exists for each $n$ a closed set $F_n \subseteq I \times I$ such that $F_n \cap \text{Gr}(f \mid A_n) = \emptyset$ and $m(F_n) \leq 1 - \varepsilon_n$ for all $x \in I$. The set $F = \cap F_n$ has the required properties.

**Theorem 3.** There exists a function $f : I \times I$ of Baire class 1 such that $I$ cannot be partitioned into countably many sets $A_n$ with each restriction $f \mid A_n$ continuous.

**Proof.** In view of Theorem 2, it is sufficient for $f$ to have the property that $F \cap \text{Gr}(f) \neq \emptyset$ for every closed set $F \subseteq I \times I$ which satisfies $F_x \neq \emptyset$ for all $x \in I$. It is known [3] that there exists a function with $G_\delta$ graph having the stated property; this is not quite enough, but the example constructed explicitly in [4] is lower semicontinuous and therefore in the first Baire class.

**Note added 13 January 1973.** In a paper by L. Keldysh (Sur les fonctions premières mesurables $B$, Dokl. Akd. Nauk SSSR (N.S.) 5 (1934), 192–197) it was shown that for every $\alpha$ there exists a function $f : I \to I$ of Baire class $\alpha$, such that $I$ cannot be partitioned into countably many sets $A_n$ with each restriction $f \mid A_n$ of class less than $\alpha$, thereby answering a question of P. S. Novikov, who had already proved the result stated above as Theorem 3.

**References**


*Author's address:* Dept. of Mathematics, The University, Leicester. LE1 7RH. England.