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ON CUBES AND DICHOTOMIC TREES

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The notion of the n-cube $Q_n$ (and other notions not defined here) can be found in BEHZAD and CHARTRAND [1] or in HARARY [2]. The complete dichotomic tree $D_n$ can be defined as follows: if $n = 1$, then $D_n$ is the complete bigraph $K(1, 2)$; if $n \geq 2$, then $D_n$ is the tree obtained from two disjoint copies $T$ and $T'$ of $D_{n-1}$ and from a new vertex $v$ in such a way that $v$ is joined by one edge to the only vertex of degree 2 of $T$ and by another edge to the analogous vertex of $T'$. Thus $D_n$ has $2^n$ vertices of degree 1, one vertex of degree 2, and $2^n - 2$ vertices of degree 3. The vertex of degree 2 of $D_n$ will be referred to as its root. HAVEL and LIEBL [3] have proved that if $n \geq 2$, then $D_n$ is a subgraph of $Q_{n+2}$ but $D_n$ is not a subgraph of $Q_{n+1}$. Obviously, $D_1$ is a subgraph of $Q_2$.

If $n \geq 1$, then we denote by $\tilde{D}_n$ the tree obtained from two disjoint copies of $D_n$ in such a way that their roots are joined by an edge; this edge will be referred to as the axial edge of $\tilde{D}_n$. Obviously, $\tilde{D}_n$ has $2^{n+2} - 2$ vertices. Havel and Liebl [4] conjectured that $\tilde{D}_n$ is a subgraph of $Q_{n+2}$, for $n \geq 1$. In the present paper, this conjecture will be verified.

We introduce the graphs $Q_n^v$ and $Q_n^r$, which are certain local modifications of $Q_n$. Let $n \geq 2$; by $Q_n^v$ we denote the graph $Q_n + rt - s$, where $r$, $s$, and $t$ are such vertices of $Q_n$ that $rs$ and $st$ are distinct edges of $Q_n$; by $Q_n^r$ we denote the graph $Q_n - u - v$, where $u$ and $v$ are such vertices of $Q_n$ that $uv$ is an edge of $Q_n$. The first two theorems which will be proved in the present paper are:

**Theorem 1.** $D_n$ is a spanning subgraph of $Q_{n+1}^v$, for $n \geq 1$.

**Theorem 2.** $\tilde{D}_n$ is a spanning subgraph of $Q_{n+2}^r$, for $n \geq 1$.

Both theorems will be easily obtained from the following lemma. An edge of a tree $T$ incident with an end-vertex of $T$ will be referred to as an end-edge. Let $n \geq 1$. By $\tilde{D}_n$ or $\tilde{D}_n$ we denote the tree obtained from $D_n$ by inserting two new vertices of
degree 2 into the axial edge or into one end-edge, respectively. The path of \( \tilde{D}_n \) obtained from the axial edge of \( \tilde{D}_n \) is referred to as the axial path of \( \tilde{D}_n \).

**Lemma.** \( \tilde{D}_n \) and \( \tilde{D}_n \) are spanning subgraphs of \( Q_{n+2} \), for \( n \geq 1 \).

**Proof.** Obviously, the graphs \( \tilde{D}_n, \tilde{D}_n \) and \( Q_{n+2} \) have the same number of vertices. Hence it is sufficient to prove that both \( \tilde{D}_n \) and \( \tilde{D}_n \) are subgraphs of \( Q_{n+2} \).

Let \( m \) be a positive integer. We shall say that a tree \( T \) is \( m \)-valued if each edge of \( T \) is assigned a positive integer not exceeding \( m \). As follows from the work of 
\textsc{Havel} and \textsc{Morávek} [5], a tree \( T \) is a subgraph of \( Q_m \) if and only if \( T \) can be \( m \)-valued so that

(1) for each path \( P \) of \( T \), there exists \( k \) such that precisely an odd number of edges belonging to \( P \) is assigned \( k \).

(Cf. also \textsc{Hlavíčka} [6].)

(A) We shall prove that \( \tilde{D}_n \) can be \( (n + 2) \)-valued so that (1) holds and that the edges of the axial path are assigned the integers 1, \( n + 1 \), and \( n + 2 \) (in some order).

The case \( n = 1 \) is obvious. The case \( n = 2 \) is given in Fig. 1.

![Fig. 1.](image)

Let \( n = m \geq 3 \). Assume that for \( n = m - 2 \), the statement is proved. Consider four disjoint copies of \( \tilde{D}_{n-2} \) which are \( n \)-valued so that (1) holds and that they can be expressed as in Fig. 2, where \( R_i \) and \( R'_i \) are \( n \)-valued copies of \( D_{n-2} \). If we identify the root of each of the \( n \)-valued trees \( R_i \) and \( R'_i \) with the vertex \( r_i \) and \( r'_i \), respectively, in Fig. 3, we obtain an \( (n + 2) \)-valued tree \( \tilde{D}_n \). Obviously, the edges of the axial

![Fig. 2.](image)
path are assigned 1, \( n + 1 \), and \( n + 2 \). It is routine to prove that this valuation fulfils (1).

(B) Let \( n \geq 1 \); by \( D^*_n \) we denote the tree obtained from \( D_n \) by inserting two new vertices of degree 2 into one end-edge of \( D_n \); the vertex of \( D^*_n \) obtained from the root of \( D_n \) will be referred to as the root of \( D^*_n \). We shall prove that \( D_n \) can be \((n + 2)\)-valued so that (1) holds. The case \( n = 1 \) is obvious. Let \( n = m \geq 2 \). Assume that for \( n = m - 1 \), the statement is proved. Consider disjoint \( D_{n-1} \) and \( D_{n-1} \) which are \((n + 1)\)-valued so that (1) holds and that they can be expressed as in Fig. 4, where \( T_1 \) and \( T_2 \) are \((n + 1)\)-valued copies of \( D_{n-1} \), and \( T'_2 \) is an \((n + 1)\)-valued copy of \( D^*_{n-1} \). Join the root of \( T_2 \) by an edge assigned \( n + 2 \) to the vertex \( t_2 \) and the root of \( T'_2 \) by an edge assigned \( n + 2 \) to the vertex \( t'_2 \) (see Fig. 5). Thus we obtain \( D_n \) which is \((n + 2)\)-valued such that (1) holds. Hence the lemma follows.

Proof of Theorem 1. The case \( n = 1 \) is obvious. Let \( n \geq 2 \) and let \( t, u, v \) and \( w \) be such vertices of \( D_{n-1} \) that \( tu, uv \) and \( vw \) are the edges of the axial path. Then \( D_n = D_{n-1} + uw - v \). Thus the lemma implies the theorem.

Proof of Theorem 2 directly follows from the lemma.

Corollary. \( D_n \) is a subgraph of \( Q_{n+2} \), for \( n \geq 1 \).

Let \( n \geq 2 \). By \( D_n \) we denote the tree obtained from disjoint \( D_{n-1} \) and \( D_n \) by joining their roots by an edge. As \( D_n \) is a subgraph of \( D_n \), it is also a subgraph of \( Q_{n+2} \).
It has been pointed out by Havel and Liebl [4] that the trees $D_n$ and $\tilde{D}_n$ are useful for a study of trees with the maximum degree 3.

**Theorem 3.** Let $T$ be a tree with the diameter $d \geq 2$ and with the maximum degree 3. Then $T$ is a subgraph of $Q_{\lfloor d/2 \rfloor + 2}$.

Proof. The case $d = 2$ is obvious. Let $d = 2n$, $n \geq 2$; it is easily seen that $T$ is a subgraph of $D_n$ and thus $T$ is a subgraph of $Q_{n+2}$. Let $d = 2n + 1$, $n \geq 1$; then $T$ is a subgraph of $\tilde{D}_n$ and thus $T$ is a subgraph of $Q_{n+2}$. Hence the theorem follows.

References


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