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Tolerance relation on lattices

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E. C. Zeeman [3] has defined the tolerance as a binary relation on a set, which is reflexive and symmetric. M. Arbib [1, 2] has applied this concept to the theory of automata. In [4] and [5], tolerances compatible with algebraic structures are studied.

Let $A = \langle A, \mathcal{F} \rangle$ be an algebraic structure with the element set $A$ and the set of operations $\mathcal{F}$. Let $\xi$ be a tolerance on $A$. The tolerance $\xi$ is compatible with $A$, if and only if for any $n$-ary operation $f \in \mathcal{F}$, where $n$ is a positive integer, and for any $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ of $A$ such that $(x_i, y_i) \in \xi$ for $i = 1, \ldots, n$ we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi$.

Here we shall study tolerances which are compatible with lattices. Some simple results in this topic are in [4]. From the definition of a tolerance compatible with an algebraic structure it follows that a tolerance $\xi$ is compatible with a lattice $L$, if and only if for any four elements $x_1, x_2, y_1, y_2$ of $L$ such that $(x_1, x_2), (y_1, y_2) \in \xi$ we have $(x_1 \land x_2, y_1 \land y_2) \in \xi, (x_1 \lor x_2, y_1 \lor y_2) \in \xi$.

**Theorem 1.** Let $L$ be a lattice, let $\xi$ be a tolerance compatible with $L$. Let $(a, b) \in \xi$ for some $a \in L, b \in L$. Then for any $x$ and $y$ from interval $\langle a \land b, a \lor b \rangle$ we have $(x, y) \in \xi$.

**Proof.** From $(a, b) \in \xi, (b, b) \in \xi$ ($\xi$ is reflexive) we obtain $(a \land b, b \land b) = (a \land b, b) \in \xi, (a \lor b, b \lor b) = (a \lor b, b) \in \xi$. Analogously we obtain $(a \land b, a) \in \xi, (a \lor b, a) \in \xi$. Further from $(a \land b, a) \in \xi$ and $(a \land b, b) \in \xi$ we obtain $((a \land b) \lor (a \land b), a \lor b) = (a \land b, a \lor b) \in \xi$. Now let $x \in \langle a \land b, a \lor b \rangle, y \in \langle a \land b, a \lor b \rangle$. From $(a \land b, a \lor b) \in \xi, (x, x) \in \xi$ we have $((a \land b) \lor x, a \lor b \lor x) = (x, a \lor b) \in \xi$ and analogously $(y, a \lor b) \in \xi$. Taking meets, from $(x, a \lor b) \in \xi, (a \lor b, y) \in \xi$ we obtain $(x \land (a \lor b), y \land (a \lor b)) = (x, y) \in \xi$. As $x$ and $y$ were chosen arbitrarily, this holds for any two elements of the interval $\langle a \land b, a \lor b \rangle$. 

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Corollary. In a lattice $L$ with $O$ and $I$, for any tolerance $\xi$ compatible with $L$ the following three assertions are equivalent:

(i) For some $a \in L$ there exists a complement $a'$ and $(a, a') \in \xi$.
(ii) $(O, I) \in \xi$.
(iii) $\xi$ is the universal relation on $L$.

$O$ and $I$ denote respectively the least and the greatest element of the lattice.

Theorem 2. Let $B$ be a Boolean algebra, let $\xi$ be a tolerance compatible with the operations of join and meet in $B$. Then $\xi$ is a congruence on $B$.

Remark. Here we do not suppose a priori that $\xi$ is compatible with the complementation, but this follows from the assertion.

Proof. Let $B_0$ be the set of all elements $x \in B$ such that $(x, O) \in \xi$. If $x \in B_0$, $y \in B$, then $x \wedge y \in B_0$, because $(x, O) \in \xi$, $(y, y) \in \xi$ implies $(x \wedge y, O) \in \xi$. Therefore $B_0$ is an ideal of $B$. Any ideal of a Boolean algebra determines uniquely a congruence on it. Let $\kappa$ be the congruence determined on $B$ by $B_0$. We shall prove $\kappa \subset \xi$. If $a \in B_0$, $b \in B_0$, then $(a, O) \in \xi$, $(O, b) \in \xi$ and this implies $(a, b) \in \xi$. If $c, d$ are elements of the same congruence class of $\kappa$, then $c = a \vee z$, $d = b \vee z$, where $a \in B_0$, $b \in B_0$, $z \in B$. From $(a, b) \in \xi$, $(z, z) \in \xi$ we obtain $(a \vee z, b \vee z) = (c, d) \in \xi$. Therefore $\kappa \subset \xi$. Now let $(u, v) \in \xi$, let $\bar{v}$ be the complement of $v$. From $(u, v) \in \xi$, $(\bar{v}, \bar{v}) \in \xi$ we obtain $(u \wedge \bar{v}, v \wedge \bar{v}) = (u \wedge \bar{v}, O) \in \xi$ and $u \wedge \bar{v} \in B_0$. This means that the class of $\kappa$ containing $u$ is the complement of the class of $\kappa$ containing $\bar{v}$ in the Boolean factor-algebra $B/\kappa$. But obviously also the class of $\kappa$ containing $v$ is the complement of the class of $\kappa$ containing $\bar{v}$. As $B/\kappa$ is also a Boolean algebra, this complement is unique and $u$ and $v$ belong to the same congruence class of $\kappa$. We have proved $\xi \subset \kappa$ and therefore $\xi = \kappa$.

Theorem 3. Let $C$ be a chain with at least three elements. Then there exist a tolerance $\xi$ compatible with $C$ which is not a congruence.

Proof. Choose three elements $a, b, c$ of $L$ so that $a < b < c$. Now let $\xi$ consist of all pairs $(x, y)$, where either both $x$ and $y$ belong to $\langle a, b \rangle$, or both $x$ and $y$ belong to $\langle b, c \rangle$, or $x = y$. This is evidently a tolerance on $C$. Now let $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$. If all elements $x_1, y_1, x_2, y_2$ belong to $\langle a, b \rangle$, then also $x_1 \wedge x_2, x_1 \vee x_2, y_1 \wedge y_2, y_1 \vee y_2$ belong to $\langle a, b \rangle$, because the interval $\langle a, b \rangle$ is a sublattice of $C$; then $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. We proceed analogously if all elements $x_1, y_1, x_2, y_2$ belong to $\langle b, c \rangle$. If $x_1, y_1$ belong to $\langle a, b \rangle$ and $x_2, y_2$ belong to $\langle b, c \rangle$, then $x_1 \wedge x_2 = x_1, x_1 \vee x_2 = x_2, y_1 \wedge y_2 = y_1, y_1 \vee y_2 = y_2$, therefore $x_1 \wedge x_2, y_1 \wedge y_2$ belong to $\langle a, b \rangle$, $x_1 \vee x_2, y_1 \vee y_2$ belong to $\langle b, c \rangle$ and again $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi$. If $x_1$ belongs neither to $\langle a, b \rangle$, nor to $\langle b, c \rangle$, then necessarily $x_1 = y_1$. If it is less than $a$ and $x_2, y_2$ belong both to $\langle a, b \rangle$ or both to $\langle b, c \rangle$, we have $x_1 \wedge x_2 = x_1, y_1 \wedge y_2 = y_1, x_1 \vee x_2 = x_2$. 395
y_1 \lor y_2 = y_2 and again \((x_1 \land x_2, y_1 \land y_2) \in \xi, (x_1 \lor x_2, y_1 \lor y_2) \in \xi\). The same follows analogously if \(x_1 = y_1 > c\) and \(x_2, y_2\) belong either both to \(\langle a, b \rangle\), or both to \(\langle b, c \rangle\). Finally, if \(x_1 = y_1, x_2 = y_2\), the proof is easy. We have obtained that \(\xi\) is a tolerance compatible with \(C\). We have \((a, b) \in \xi, (b, c) \in \xi, \) but \((a, c) \notin \xi\) and \(\xi\) is not a congruence.

**Theorem 4.** There exists a non-complete distributive lattice \(L\) such that any tolerance compatible with \(L\) is a congruence.

**Proof.** Let \(M\) be a set of cardinality \(\aleph_0\), let \(L\) be the lattice of all finite subsets of \(M\) ordered by set inclusion. The elements of \(L\) will be denoted by capital letters as sets. Let \(\xi\) be a tolerance compatible with \(L\), let \(A, B, C\) be three elements of \(L\) such that \((A, B) \in \xi, (B, C) \in \xi\). Let \(M_0 = A \cup B \cup C;\) it is a finite set. Let \(L_0\) be the lattice of all subsets of \(M_0;\) it is a Boolean algebra and a sublattice of \(L\). Let \(\xi_0\) be the restriction of \(\xi\) onto \(L_0\). Then \(\xi_0\) is a tolerance compatible with \(L_0;\) as \(L_0\) is a Boolean algebra, \(\xi_0\) is a congruence on \(L_0\) and \((A, C) \in \xi_0\). But as \(\xi_0 \subset \xi\), we have also \((A, C) \in \xi\). As \(A, B, C\) and \(\xi\) were chosen quite arbitrarily, any tolerance compatible with \(L\) is transitive, therefore it is a congruence. The lattice \(L\) is evidently distributive and non-complete.

**Theorem 5.** There exists a non-complete distributive lattice in which a tolerance \(\xi\) exists which is not a congruence and is compatible with \(L\).

**Proof.** We shall construct \(L\). The vertices of \(L\) are ordered pairs of integers and \([x_1, y_1] \leq [x_2, y_2]\) if and only if simultaneously \(x_1 \leq x_2, y_1 \leq y_2\). Evidently

\[
[x_1, y_1] \land [x_2, y_2] = [\min (x_1, x_2), \min (y_1, y_2)],
\]

\[
[x_1, y_1] \lor [x_2, y_2] = [\max (x_1, x_2), \max (y_1, y_2)].
\]

We define \(\xi\) so that \(([x_1, y_1], [x_2, y_2]) \in \xi,\) if and only if simultaneously \(\vert x_1 - x_2 \vert \leq 1, \vert y_1 - y_2 \vert \leq 1\). It is evidently a tolerance. Now let \(([x_1, y_1], [x_2, y_2]) \in \xi, ([u_1, v_1], [u_2, v_2]) \in \xi).\) We shall prove that then also \(([x_1, y_1] \land [u_1, v_1], [x_2, y_2] \land [u_2, v_2]) \in \xi,\) this means \(\vert \min (x_1, u_1) - \min (x_2, u_2) \vert \leq 1\) and \(\vert \min (y_1, v_1) - \min (y_2, v_2) \vert \leq 1.\) If \(x_1 \leq u_1, x_2 \leq u_2,\) then \(\min (x_1, u_1) = x_1, \min (x_2, u_2) = x_2,\) and we have \(\vert x_1 - x_2 \vert \leq 1,\) because \(([x_1, y_1], [x_2, y_2]) \in \xi).\) If \(x_1 \geq u_1, x_2 \geq u_2,\) then \(\min (x_1, u_1) = u_1, \min (x_2, u_2) = u_2\) and the situation is similar. Now let \(x_1 \leq u_1, x_2 \geq u_2.\) Then \(x_1 - x_2 \leq x_1 - u_2 \leq u_1 - u_2.\) But \(\vert x_1 - x_2 \vert \leq 1, \vert u_1 - u_2 \vert \leq 1,\) therefore \(x_1 - x_2 \geq -1, u_1 - u_2 \leq 1\) and thus \(-1 \leq x_1 - u_2 \leq 1,\) which means \(\vert x_1 - u_2 \vert \leq 1.\) Analogously we proceed in the case \(x_1 \geq u_1, x_2 \leq u_2.\) We have proved that \(\vert \min (x_1, u_1) - \min (x_2, u_2) \vert \leq 1.\) The proof of the inequality \(\vert \min (y_1, v_1) - \min (y_2, v_2) \vert \leq 1\) is quite analogous. Thus \(([x_1, y_1] \lor [u_1, v_1], [x_2, y_2] \lor [u_2, v_2]) \in \xi,\) Dually we can prove also \(([x_1, y_1] \land [u_1, v_1], [x_2, y_2] \land [u_2, v_2]) \in \xi,\) and therefore \(\xi\) is a tolerance compatible with \(L\). We have \(([0, 0], [1, 1]) \in \xi, ([1, 1], [2, 2]) \in \xi,\) but \(([0, 0], [2, 2]) \notin \xi\) and hence \(\xi\) is not transitive.
Theorem 6. For each cardinal number $n \geq 5$ there exists a modular non-distributive lattice $L$, $|L| = n$, such that any tolerance $\xi$ compatible with it is either the identity (i.e., $(x, y) \in \xi$ if and only if $x = y$), or the universal relation (i.e., $(x, y) \in \xi$ for each $x$ and $y$).

Proof. Let $K$ be a set of cardinality $n - 2$, if $n$ is finite, and of the cardinality $n$, if $n$ is infinite. The set of elements of $L$ consists of the elements $a_k$ ($k \in K$) and of the elements $0, 1$. We define $0 < a_k < 1$ for all $k \in K$ and $a_k \parallel a_l$ for $k \in K$, $l \in K$, $k \neq l$. Let $\xi$ be a tolerance compatible with $L$ and suppose that there exist $x \in L$, $y \in L$ such that $x \neq y$, $(x, y) \in \xi$. As $\xi$ is symmetric, we may suppose without loss of generality that either $x \leq y$, or $x \parallel y$. According to Theorem 1 it suffices to prove that then $(O, I) \in \xi$. If $x = O$, $y = I$, this is immediate. If $x = a_k$, $y = a_l$ for $k \in K$, $l \in K$, $k \neq l$ then according to Corollary, $\xi$ is the universal relation, because $a_j$ is a complement of $a_k$. If $x = O$, $y = a_k$ for some $k \in K$, then take some $a_l$ for $l \in K$, $l \neq k$; as $|L| \geq 5$, such $a_l$ exists. From $(O, a_k) \in \xi$, $(a_k, a_l) \in \xi$ we obtain $(O \lor a_l, a_k \lor a_l) = = (a_l, I) \in \xi$. If we take some $m \in K$, $m \neq k$, $m \neq l$, we can prove in the same way that $(a_m, I) \in \xi$. From $(a_l, I) \in \xi$, $(a_m, I) \in \xi$ we obtain $(a_l \land a_m, I \land I) = (O, I) \in \xi$. In the case $x = a_k$, $y = I$ we proceed dually.

Remark. For $n = 5$ this lattice is actually the “forbidden sublattice” for distributive lattices.

Theorem 7. There exists a non-modular lattice on which a tolerance compatible with it exists which is not a congruence.

\begin{center}
\begin{tikzpicture}
  \node (O) at (0,0) {$O$};
  \node (a) at (1,-1) {$a$};
  \node (b) at (2,0) {$b$};
  \node (c) at (1,1) {$c$};
  \node (d) at (2,1.5) {$d$};
  \node (e) at (1,3) {$e$};
  \node (f) at (3,1.5) {$f$};

  \draw (O) -- (a) -- (b) -- (c) -- (O);
  \draw (O) -- (d) -- (e) -- (f) -- (O);
  \draw (a) -- (d);
  \draw (b) -- (e);
  \draw (c) -- (f);
\end{tikzpicture}
\end{center}

Proof. The Hasse diagram of such a lattice is in Fig. 1. The tolerance $\xi$ is given so that $(x, y) \in \xi$, if and only if $x$ and $y$ lie simultaneously either in $<O, b>$, or in $<a, d>$, or in $<c, f>$, or in $<e, I>$. The reader may verify himself that $\xi$ is compatible with $L$. The tolerance $\xi$ is not a congruence.
Theorem 8. Let $L$ be a lattice, $L_0$ its sublattice, let there exist a homomorphism $\varphi$ which maps $L$ onto a lattice $L_1$ and such that $\varphi(x) = \varphi(y)$, if and only if $x \in L_0$, $y \in L_0$. On $L_0$ let there exist a tolerance $\xi_0$ compatible with $L_0$ which is not a congruence. Then there exists a tolerance $\xi$ compatible with $L$ which is not a congruence.

Proof. Let $\xi$ consist of all pairs of elements which are in $\xi_0$ and of all pairs of equal elements of $L$. We shall prove that $\xi$ is compatible with $L$. Let $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$. If all elements $x_1$, $y_1$, $x_2$, $y_2$ belong to $L_0$, then $(x_1, y_1) \in \xi_0$, $(x_2, y_2) \in \xi_0$. The elements $x_1 \wedge x_2$, $x_1 \vee x_2$, $y_1 \wedge y_2$, $y_1 \vee y_2$ belong to $L_0$ and $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi_0 \subseteq \xi$, $(x_1 \vee x_2, y_1 \vee y_2) \in \xi_0 \subseteq \xi$. Now let $x_1 \in L_0$, $x_2 \in L_0$, $x_2 = y_2 \notin L_0$. If $x_2 \leq x_1$, then $\varphi(x_2) \leq \varphi(x_1) = \varphi(y_1)$ and therefore $y_2 = x_2 \leq y_1$. We have $(x_1 \wedge x_2, y_1 \wedge y_2) = (x_2, y_2) \in \xi$, $(x_1 \vee x_2, y_1 \vee y_2) = (x_1, y_1) \in \xi$. In the case $x_2 \geq x_1$, we proceed dually. If $x_1 \parallel x_2$, we have $\varphi(x_1) \parallel \varphi(x_2)$, because evidently $\varphi(x_1) \neq \varphi(x_2)$. But $\varphi(x_1) = \varphi(y_1)$, therefore $\varphi(x_2) \parallel \varphi(y_1)$ in $L_1$ and $x_2 \parallel y_1$ in $L$.

In $L_1$ we have $\varphi(x_2) \wedge \varphi(x_1) = \varphi(x_2) \wedge \varphi(y_1) \neq \varphi(x_1)$, therefore $x_2 \wedge y_1 \notin L_0$, $x_2 \wedge y_1 \notin L_0$. But as $\varphi(x_2 \wedge x_1) = \varphi(x_2) \wedge \varphi(x_1) = \varphi(x_2) \wedge \varphi(y_1) = \varphi(x_2 \wedge y_1)$, the elements $x_2 \wedge x_1$, $x_2 \wedge y_1$ must be equal (they are not in $L_0$ and their images in $\varphi$ are equal). Thus $(x_1 \wedge x_2, y_1 \wedge y_2) \in \xi$. For joins we proceed dually. Finally, if $x_1 = y_1$, $x_2 = y_2$, the proof is easy. We have proved that $\xi$ is a tolerance compatible with $L$. Now if $\xi_0$ is not transitive, also $\xi$ is not transitive, because $\xi$ contains no pair of elements of $L_0$ which are not in $\xi_0$.

In the end we shall prove a theorem concerning tolerance relations on arbitrary algebraic structures.

Theorem 9. Let $A = \langle A, \mathcal{F} \rangle$ be an algebraic structure. The tolerances compatible with $A$ form a lattice $LT(A)$ with respect to the set inclusion. In general, this lattice is not a sublattice (in the algebraic sense) of the lattice of all tolerances on $A$.

Proof. As shown in [5], the intersection of two tolerances compatible with $A$ is a tolerance compatible with $A$. Thus in $LT(A)$ we put $\xi_1 \wedge \xi_2 = \xi_1 \cap \xi_2$ for any two tolerances $\xi_1$, $\xi_2$ which are compatible with $A$. Now consider the set of all tolerances which are compatible with $A$ and which contain $\xi_1 \cup \xi_2$. This set is non-empty, because it contains the universal relation on $A$. It is closed under intersection, the intersection of all tolerances of this set being a tolerance compatible with $A$ and containing $\xi_1 \cup \xi_2$. This tolerance will be denoted by $\xi_1 \vee \xi_2$ and it will be the join of $\xi_1$ and $\xi_2$ in $LT(A)$, because it is contained in all tolerances compatible with $A$ which contain $\xi_1 \cup \xi_2$.

In general $\xi_1 \vee \xi_2$ need not be equal to $\xi_1 \cup \xi_2$. For example, let $A$ be the lattice whose elements are $a$, $b$, $O$, $I$ and in which $O < a < I$, $O < b < I$, $a \parallel b$. Let $\xi_1 = \{(O, O), (O, a), (a, O), (a, a), (b, b), (b, I), (I, b), (I, I)\}$, $\xi_2 = \{(O, O), (O, b), (a, a), (a, I), (b, O), (b, b), (I, a), (I, I)\}$. These tolerances are compatible with $A$; the proof is left to the reader. The tolerance $\xi_1 \vee \xi_2$ is the universal relation, because
(O, 1) ∈ ξ_1 ∨ ξ_j; we obtain this from (O, a) ∈ ξ_1 ⊆ ξ_1 ∨ ξ_2, (O, b) ∈ ξ_2 ⊆ ξ_1 ∨ ξ_2 taking joins. But the set union ξ_1 ∪ ξ_2 does not contain (O, 1). Therefore LT(A) is not a sublattice in the algebraic sense of the lattice of all tolerances on A.

References


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