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Časopis pro pěstování matematiky, Vol. 99 (1974), No. 4, 400--404

Persistent URL: http://dml.cz/dmlcz/117861

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SUFFICIENT CONDITIONS FOR LOCALLY CONNECTED GRAPHS

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(Received September 21, 1973)

Let $G$ be a graph without isolated vertices, and let $v$ be a vertex of $G$. The *neighborhood* of $v$, denoted by $N(v)$, is the subgraph of $G$ induced by the set $N(v)$ of vertices of $G$ adjacent with $v$. The graph $G$ is called *locally connected* if the neighborhood of every vertex of $G$ is connected.

In [1] CHARTRAND and PIPPERT showed that if the minimum degree $\delta(G)$ of a graph $G$ of order $p$ exceeds $\frac{p}{3} - 1$, then $G$ is locally connected. More generally, it was proved in [1] that if $G$ is a graph of order $p$ such that for every pair $u, v$ of vertices, $\deg u + \deg v > \frac{p}{3} - 1$, then $G$ is locally connected. Hence, it is possible for some vertex of a graph $G$ to have degree at most $\frac{p}{3} - 1$ (with the degrees of all other vertices exceeding $\frac{p}{3} - 1$) and still be assured that $G$ is locally connected.

It is the object of this article to determine the number of vertices of specified degrees not exceeding $\frac{p}{3} - 1$ which insures that a given graph be locally connected.

The results we present are reminiscent of work on hamiltonian graphs. DIRAC [2] proved that for a graph $G$ of order $p \geq 3$, if $\delta(G) \geq p/2$, then $G$ is hamiltonian. ORE [4] extended this result by showing that if $\deg u + \deg v \geq p \geq 3$ for every pair $u, v$ of nonadjacent vertices, then $G$ is hamiltonian. PÓSA [5] then proceeded to provide a sufficient condition for hamiltonian graphs which allows even more vertices of degree less than $p/2$, including some of quite small degree.

First we show that no vertex of a graph $G$ of order $p$ can have degree much less than $\frac{p}{3} - 1$ to assure local connectedness. In this respect, it is convenient to employ the *join* $G_1 + G_2$ of two disjoint graphs $G_1$ and $G_2$, defined as that graph whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and whose edge set is

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2 | v_1 \in V(G_1), \ v_2 \in V(G_2)\}.$$ 

The *union* of graphs $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is the graph for which

$$V(G_1 \cup G_2) = V(G_1) \cup V(G_2), \quad \text{and} \quad E(G_1 \cup G_2) = E(G_1) \cup E(G_2).$$ 

*) This work was supported in part by an NSF Science Faculty Fellowship.
The union of \( n \) graphs, each of which is isomorphic to \( G \), is denoted by \( nG \); if \( G \) is connected, the graph \( nG \) has \( n \) components, each of which is isomorphic to \( G \).

As usual, \( \{ \} \) denotes the least integer function in what follows. All definitions and notation not given here may be found in [3].

**Proposition.** Let \( G \) be a graph of order \( p \geq 5 \). If \( G \) has one vertex of degree \( 2\{\frac{1}{3}(p - 1)\} - 2 \) and all others have degree exceeding \( \frac{2}{3}(p - 1) \), then \( G \) need not be locally connected.

**Proof.** Let \( k = \{\frac{1}{3}(p - 1)\} \) and consider the graph \( G = 2K_{k-1} + \left( \{v\} \cup K_{p+1-2k} \right) \). Then \( \deg v = 2\{\frac{1}{3}(p - 1)\} - 2 \), and all other vertices have degree exceeding \( \frac{2}{3}(p - 1) \). Since \( \langle N(v) \rangle \) is disconnected, \( G \) is not locally connected.

Thus, by the preceding proposition, we may not allow even a single vertex to have degree as small as \( 2\{\frac{1}{3}(p - 1)\} - 2 \) (with all other degrees exceeding \( \frac{2}{3}(p - 1) \)) and be assured that the graph is locally connected. In the case of vertices of degree \( 2\{\frac{1}{3}(p - 1)\} - 1 \), we have the following result.

**Theorem 1.** Let \( G \) be a graph of order \( p \) which has up to \( 2\{\frac{1}{3}(p - 1)\} - p - 1 \) vertices of degree \( 2\{\frac{1}{3}(p - 1)\} - 1 \) and all others of degree greater than \( \frac{2}{3}(p - 1) \). Then \( G \) is locally connected.

**Proof.** If \( p \equiv 2 \pmod{3} \), then \( 2\{\frac{1}{3}(p - 1)\} - 1 > \frac{2}{3}(p - 1) \), so \( \delta(G) > \frac{2}{3}(p - 1) \). Thus, \( G \) is locally connected.

For \( p \equiv 0 \pmod{3} \) or \( p \equiv 1 \pmod{3} \), suppose \( G \) is not locally connected. Let \( v \) be a vertex of \( G \) such that \( \langle N(v) \rangle \) is not connected.

**Case 1.** Suppose \( \deg v = 2\{\frac{1}{3}(p - 1)\} - 1 \).

Let \( G_1 \) be a component of \( \langle N(v) \rangle \) of minimum order, say \( |V(G_1)| = r \). Then \( r \leq 2\{\frac{1}{3}(p - 1)\} - 1 \), so \( r \leq \{\frac{1}{3}(p - 1)\} - 1 \). If \( u \in V(G_1) \), then \( \deg u \leq r + p - 2\{\frac{1}{3}(p - 1)\} \leq p - 1 - \{\frac{1}{3}(p - 1)\} \leq \frac{2}{3}(p - 1) \). Thus each vertex of \( G_1 \) has degree at most \( \frac{2}{3}(p - 1) \), so the degree of each vertex of \( G_1 \) must be \( 2\{\frac{1}{3}(p - 1)\} - 1 \). Therefore, \( r \leq 2\{\frac{1}{3}(p - 1)\} - p - 2 \) since there are at most \( 2\{\frac{1}{3}(p - 1)\} - p - 1 \) vertices of degree \( 2\{\frac{1}{3}(p - 1)\} - 1 \), one of which is \( v \). Hence \( \deg u \leq r + p - 2\{\frac{1}{3}(p - 1)\} \leq 2\{\frac{1}{3}(p - 1)\} - 1 - \{\frac{1}{3}(p - 1)\} - 2 = 2\{\frac{1}{3}(p - 1)\} - 2 \), since \( p \not\equiv 1 \pmod{3} \). By hypothesis this is impossible so Case 1 cannot happen.

**Case 2.** Suppose \( \deg v = k > \frac{2}{3}(p - 1) \).

Select \( G_1 \) as in Case 1 so that \( r \leq k/2 \). If \( u \in V(G_1) \), then \( \deg u \leq r + p - 1 - k < \frac{2}{3}(p - 1) \). Thus \( r \leq 2\{\frac{1}{3}(p - 1)\} - p - 1 \), so \( \deg u \leq r + p - 1 - k < 2\{\frac{1}{3}(p - 1)\} - \frac{2}{3}(p - 1) - 2 < 2\{\frac{1}{3}(p - 1)\} - 1 \). But, by hypothesis, this is impossible, so Case 2 cannot happen.

The following example shows that the result in Theorem 1 is sharp.
Example 1. Let $G = (K_{2k - p - 1} \cup K_{p - k}) + \{v\} \cup K_{p - k}$, where $p \geq 7$ and $k = \{i(p - 1)\}$. Then $G$ has $2\{\frac{3}{p}(p - 1)\} - p$ vertices of degree $\frac{3}{p}(p - 1) - 1$ and all other vertices have degrees exceeding $\frac{3}{p}(p - 1)$. Since $\langle N(v) \rangle$ is disconnected, $G$ is not locally connected.

As we noted at the beginning of the proof of Theorem 1, when $p \equiv 2 \pmod{3}$, then $\delta(G) > \frac{3}{p}(p - 1)$. However, if $p \equiv 2 \pmod{3}$, then $2\{\frac{3}{p}(p - 1)\} - 2 < \frac{3}{p}(p - 1)$ and all other vertices have degree exceeding $\frac{3}{p}(p - 1)$. Thus, by the Proposition, when $p \equiv 2 \pmod{3}$, if $G$ has as few as one vertex of degree not exceeding $\frac{3}{p}(p - 1)$, then $G$ need not be locally connected.

If $p \equiv 0 \pmod{3}$, then by Theorem 1, $G$ may have as many as $2\{\frac{3}{p}(p - 1)\} - p - 1$ vertices of degree $2\{\frac{3}{p}(p - 1)\} - 1$ and all others of degree greater than $\frac{3}{p}(p - 1)$, and necessarily $G$ is locally connected. Now when $p \equiv 0 \pmod{3}$, we have $2\{\frac{3}{p}(p - 1)\} - 1 = \{i(p - 1)\} - 1$, so Theorem 1 is best possible.

The remaining case to consider is $p \equiv 1 \pmod{3}$. In this case, Theorem 1 states that if $G$ has a certain number of vertices of degree $2\{\frac{3}{p}(p - 1)\} - 1 = \frac{3}{p}(p - 1) - 1$ and all others have degree exceeding $\frac{3}{p}(p - 1)$, then $G$ must be locally connected. We next determine what combination of vertices of degrees $\frac{3}{p}(p - 1) - 1$ and $\frac{3}{p}(p - 1)$, with all other vertices having degree exceeding $\frac{3}{p}(p - 1)$, insures that $G$ is locally connected.

Theorem 2. Let $p \equiv 1 \pmod{3}$ and let $k$ be such that $0 < k < \frac{3}{p}(p - 1) - 1$. If a graph $G$ has $k$ vertices of degree $\frac{3}{p}(p - 1)$ and $\frac{3}{p}(p - 1) - 1 - k$ vertices of degree $\frac{3}{p}(p - 1) - 1$, with all other vertices of degree exceeding $\frac{3}{p}(p - 1)$, then $G$ is locally connected.

Proof. Assume $G$ is not locally connected and let $v$ be a vertex of $G$ for which $\langle N(v) \rangle$ is not connected. We consider three cases determined by the degree of $v$.

Case 1. Suppose $\deg v = \frac{3}{p}(p - 1) - 1$.

Let $G_1$ be a component of $\langle N(v) \rangle$ of smallest order, say $|V(G_1)| = r$. Thus, $r \leq \frac{3}{p}(p - 1) - 1$, since $r$ is an integer and $p \equiv 1 \pmod{3}$. Let $u \in V(G_1)$. Then $\deg u \leq r + p - \frac{3}{p}(p - 1) \leq \frac{3}{p}(p - 1)$. Thus each vertex in $G_1$ has degree $\frac{3}{p}(p - 1) - 1$ or $\frac{3}{p}(p - 1)$, and since there are $\frac{3}{p}(p - 1) - 1$ such vertices, one of which is $v$, necessarily $r \leq \frac{3}{p}(p - 1) - 2$. Thus $\deg u \leq \frac{3}{p}(p - 1) - 2$. But $G$ contains $\frac{3}{p}(p - 1) - 1 - k$ vertices of degree $\frac{3}{p}(p - 1) - 1$, one of which is $v$, so $r \leq \frac{3}{p}(p - 1) - 2 - k$. Therefore, $\deg u \leq \frac{3}{p}(p - 1) - 2 - k$. But $k > 0$, so $\deg u \leq \frac{3}{p}(p - 1) - 2$, which by hypothesis is impossible. Thus Case 1 cannot happen.

Case 2. Suppose $\deg v = \frac{3}{p}(p - 1)$.

Let $G_1$ and $r$ be as in Case 1. Then $r \leq \frac{3}{p}(p - 1)$. If $u \in V(G_1)$, then $\deg u \leq r + \frac{3}{p}(p - 1) \leq \frac{3}{p}(p - 1)$. But $G$ has $\frac{3}{p}(p - 1)$ such vertices, one of which is $v$, so $r \leq \frac{3}{p}(p - 1) - 2$. Hence $\deg u \leq \frac{3}{p}(p - 1) - 2$, which by hypothesis is impossible. Thus Case 2 cannot occur.
Case 3. Suppose \( \deg v = t > \frac{1}{3}(p - 1) \).

Let \( G_1 \) and \( r \) be as in Case 1, so \( r \leq t/2 \). For \( u \in V(G_1) \), \( \deg u \leq r + p - 1 - t < \frac{1}{3}(p - 1) \). Since \( G \) has \( \frac{1}{3}(p - 1) - 1 - k \) such vertices, we must have \( r \leq \frac{1}{3}(p - 1) - 1 - k \). But then \( \deg u \leq \frac{1}{3}(p - 1) - 1 - k + (p - 1) - t < \frac{1}{3}(p - 1) - 1 - k \). Since \( k > 0 \), necessarily \( \deg u < \frac{1}{3}(p - 1) - 2 \), which is impossible. Thus Case 3 is also impossible, so the assumed graph \( G \) cannot exist; that is, the theorem is valid.

An example will illustrate the sharpness of Theorem 2.

**Example 2.** Let \( p \equiv 1 \pmod{3} \), and let \( k \) satisfy \( 0 < k < \frac{1}{3}(p - 1) - 1 \). Then let \( G' = (G_1' \cup G_2') + (\{v\} \cup G_3') \), where \( G_1', G_2', \) and \( G_3' \) are complete graphs of orders \( \frac{1}{3}(p - 1) - 1 \), \( \frac{1}{3}(p - 1) \), and \( \frac{1}{3}(p - 1) + 1 \), respectively. A graph \( G \) is now defined. Select \( \frac{1}{3}(p - 1) - 1 - k \) vertices from \( G' \) and for each such vertex, we decrease its degree by one by deleting an incident edge which is also incident with a vertex in \( G_3' \). These deletions are performed so that no vertex in \( G_3' \) has degree decreased by more than one. This is possible since \( |V(G_3')| > |V(G_1')| \). Then \( G \) is the graph obtained from \( G' \) by removing the edges so described. Let \( G_i \ (i = 1, 2, 3) \) denote the subgraph of \( G \) corresponding to \( G_i' \). The subgraph \( G_1 \) has \( k \) vertices of degree \( \frac{1}{3}(p - 1) \) and \( \frac{1}{3}(p - 1) - 1 - k \) vertices of degree \( \frac{1}{3}(p - 1) - 1 \). All other vertices of \( G \) have degree at least \( \frac{1}{3}(p - 1) + 1 \), except that \( \deg v = \frac{1}{3}(p - 1) - 1 \). Since \( \langle N(v) \rangle \) is disconnected, \( G \) is not locally connected.

The only situation which has not been considered is when \( p \equiv 1 \pmod{3} \) and the only vertices whose degrees do not exceed \( \frac{1}{3}(p - 1) \) have degree \( \frac{1}{3}(p - 1) \).

**Theorem 3.** Let \( p \equiv 1 \pmod{3} \). If a graph \( G \) has no more than \( \frac{1}{3}(p - 1) \) vertices of degree \( \frac{1}{3}(p - 1) \), and all other vertices have degree greater than \( \frac{1}{3}(p - 1) \), then \( G \) is locally connected.

**Proof.** Suppose there is a graph \( G \) satisfying the hypothesis which is not locally connected. Then there is a vertex \( v \) of \( G \) such that \( \langle N(v) \rangle \) is not connected.

**Case 1.** Suppose \( \deg v = \frac{1}{3}(p - 1) \).

Let \( \langle N(v) \rangle = G_1 \cup G_2 \) where \( G_1 \) is a component of \( \langle N(v) \rangle \) of minimum order, say \( |V(G_1)| = r \). Then \( r \leq \frac{1}{3}(p - 1) \). If \( u \in V(G_1) \), then \( \deg u \leq r + \frac{1}{3}(p - 1) \leq \frac{1}{3}(p - 1) \). Thus each vertex of \( G_1 \) has degree \( \frac{1}{3}(p - 1) \) since no vertex of \( G \) has smaller degree. But then \( r \leq \frac{1}{3}(p - 1) \) and consequently \( |V(G_2)| = \frac{1}{3}(p - 1) \). Thus if \( y \in V(G_2) \), then \( \deg y \leq \frac{1}{3}(p - 1) \), so \( \deg y = \frac{1}{3}(p - 1) \). Therefore, all vertices of \( G_2 \) have degree \( \frac{1}{3}(p - 1) \). Also, \( \deg v = \frac{1}{3}(p - 1) \), so \( G \) contains at least \( \frac{1}{3}(p - 1) + 1 \) vertices of degree \( \frac{1}{3}(p - 1) \), which by hypothesis is impossible.

**Case 2.** Suppose \( \deg v = t > \frac{1}{3}(p - 1) \).

Let \( G_1 \) and \( r \) be as in Case 1, so \( r \leq t/2 \). If \( u \in V(G_1) \), then \( \deg u \leq r + p - 1 - t \).
< \frac{3}{2}(p - 1). But no vertex of G has degree less than \frac{3}{2}(p - 1), so Case 2 cannot happen.

Theorem 3, too, is best possible.

Example 3. Let G = 2K_r + (\{v\} \cup K_r), where r = (p - 1)/3. Then G has \frac{3}{2}(p - 1) + 1 vertices of degree \frac{3}{2}(p - 1), and all other vertices have degree exceeding \frac{3}{2}(p - 1). Since <N(v)> is disconnected, G is not locally connected.

References


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