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GENERALIZATION OF THE THEOREM ON THE ARGUMENT OF ALMOST PERIODIC FUNCTION

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In this paper we give a generalization of the following theorem for functions on connected commutative topological groups with values in commutative Banach algebras with unity:

Theorem. *Let x be a continuous complex valued function on real line R and let y be defined by: $y(t) = e^{x(t)}$ $(t \in R)$. Further let there exist $c > 0$ such that $|y(t)| \geqq c$ for $t \in R$. Then the function y is almost periodic iff the function x has the form $x(t) = iat + f(t)$, where f is almost periodic and a is real.*

Let G be a commutative topological group. We shall denote by p_t $(t \in G)$ the operator of translation, i.e., $p_t\, x(s) = x(t + s)$ $(t, s \in G)$ for any function x on G. The Banach space of bounded continuous functions on G with values in a Banach space B, equipped with the supremum norm $|\cdot|_\infty$, is denoted by $C_s(G, B)$. A function $x \in C_s(G, B)$ is called almost periodic iff the set $(p_t x;\ t \in G)$ is totally bounded in $C_s(G, B)$. The set of almost periodic functions on G into B, denoted by $AP(G, B)$, forms a closed linear subspace of $C_s(G, B)$. There exists a unique linear mapping M from $AP(G, B)$ into B, called the mean value, such that $M(p_t x) = M(x)$ $(x \in AP(G, B)$, $t \in G)$ and $M(x) \in \mathrm{cl\ co} R(x)$, where $R(x)$ is the range of the function x.

A continuous function u on G with values in B is called additive iff $u(t + s) = u(t) + u(s)$ for $t, s \in G$.

Lemma 1. *Let G be a commutative topological group and let B be a Banach space. Let x be a uniformly continuous function on G with values in B such that $p_t x - x \in AP(G, B)$ for any $t \in G$. Then the following conditions are equivalent:*

1. $x = u + y$, where u is additive and $y \in C_s(G, B)$.
2. There exists $c < \infty$ and a finite set $K = (t_1, \ldots, t_k) \subset G$ such that $\inf_{z \in B} \inf_{j = 1, \ldots, k} \sup_{s \in G} |p_t\, x(s) - p_{t_j}\, x(s) - z| \leqq c$ for $t \in G$.

Moreover, if the function x satisfies the condition 1 then the additive function u is uniquely determined, $u(t) = M(p_t x - x)$ *for* $t \in G$.

Proof. 1 → 2: This implication is clear.

2 → 1: Let h be defined by $h(t, s) = p_t\, x(s) - x(s) - M(p_t x - x)$ $(t, s \in G)$. Let us prove that h is bounded. Indeed, let $t \in G$ be given and let $t_j \in K$ and $z \in B$ be such that $\sup_{s\in G} |p_t\, x(s) - p_{t_j}\, x(s) - z| \leqq c + 1$. Then $|M(p_t x - p_{t_j} x) - z| \leqq c + 1$ and hence $|h(t, s)| = |p_t\, x(s) - p_{t_j}\, x(s) - z + z - M(p_t x - p_{t_j} x) + p_{t_j}\, x(s) - x(s) - M(p_{t_j} x - x)| \leqq |p_t\, x(s) - p_{t_j}\, x(s) - z| + |M(p_t x - p_{t_j} x) - z| + |p_{t_j}\, x(s) - x(s)| + |M(p_{t_j} x - x)| \leqq 2(c + 1 + \sup_{j=1,\dots,k} |p_{t_j} x - x|_\infty)$ for any $t, s \in G$.

If we set now $u(t) = M(p_t x - x)$, $y(t) = h(t, 0) + x(0)$ then u is additive, $x = u + y$ and $y \in C_s(G, B)$.

Let finally $x = u + y$, where u is additive and $y \in C_s(G, B)$. Then for any $t, s \in G$ we have $(p_t x - x)(s) = u(t) + (p_t y - y)(s)$ and hence $p_t y - y \in AP(G, B)$ and $M(p_t x - x) = u(t) + M(p_t y - y)$. For any positive integer n we have further $M(p_{nt} y - y) = \sum_{j=0}^{n-1} M(p_{jt}(p_t y - y)) = n\, M(p_t y - y)$, $|M(p_{nt} y - y)| \leqq 2|y|_\infty$ and hence $M(p_t y - y) = 0$.

Theorem 1. *Let G be a commutative topological group and let B be a Banach space. Let x be a uniformly continuous function on G with values in B such that* $p_t x - x \in AP(G, B)$ *for any* $t \in G$. *Then the following conditions are equivalent:*

1. $x = u + y$, *where u is additive and* $y \in AP(G, B)$.

2. *To any* $\varepsilon > 0$ *there exists a finite set* $K = (t_1, \dots, t_k) \subset G$ *such that* $\inf_{z\in B} \inf_{j=1,\dots,k} \sup_{s\in G} |p_t\, x(s) - p_{t_j}\, x(s) - z| \leqq \varepsilon$ *for* $t \in G$.

Proof. 1 → 2: This implication is clear.

2 → 1: By Lemma 1 the function x has the form $x = u + y$ where u is additive, $u(t) = M(p_t x - x)$ $(t \in G)$, and $y \in C_s(G, B)$. Let $\varepsilon > 0$ be given and let $K = (t_1, \dots, t_k) \subset G$ be such that $\inf_{z\in B} \inf_{j=1,\dots,k} \sup_{s\in G} |p_t\, x(s) - p_{t_j}\, x(s) - z| \leqq \frac{1}{2}\varepsilon$ for any $t \in G$. Let $t \in G$ be given and let $t_j \in K$, $z \in B$ be such that $\sup_{s\in G} |p_t\, x(s) - p_{t_j}\, x(s) - z| \leqq \frac{1}{2}\varepsilon$. Then $|M(p_t x - p_{t_j} x) - z| \leqq \frac{1}{2}\varepsilon$ and hence $|p_t y - p_{t_j} y|_\infty = \sup_{s\in G} |p_t\, x(s) - p_{t_j}\, x(s) - u(t - t_j)| \leqq \sup_{s\in G} |p_t\, x(s) - p_{t_j}\, x(s) - z| + |z - u(t - t_j)| \leqq \varepsilon$. From this it follows easily that the function y is almost periodic.

Now we shall formulate one actually known fact from the theory of commutative Banach algebras. The proof is given for completness in Appendix.

Lemma 2. *Let B be a commutative Banach algebra with unity e and let $B_0 = (x \in B;\ \exp(x) = e)$. Then for any $x, y \in B_0$, $x \neq y$, it holds $|x - y| \geqq \lg 2$. Further, to any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for any connected set $M \subset B$ for which $|\exp(x) - e| \leqq \delta(\varepsilon)$ for $x \in M$ there exists $y \in B_0$ such that $|x - y| \leqq \varepsilon$ for $x \in M$.*

Now we are able to formulate the main theorem which generalizes the classical theorem mentioned above:

Theorem 2. *Let G be a connected commutative topological group and let B be a commutative Banach algebra with unity e. Let x be a continuous function on G with values in B such that $|\exp(-x(t))| \leqq c < \infty$ for $t \in G$. Let y denote the function defined by: $y(t) = \exp(x(t))$ $(t \in G)$. Then the following conditions are equivalent:*

1. $y \in AP(G, B)$.

2. $x = x_1 + x_2$, *x_1 being additive and* $y_1, x_2 \in AP(G, B)$, *where* $y_1(t) = \exp(x_1(t))$ *for* $t \in G$.

Proof. $1 \to 2$: Let $\varepsilon > 0$ be given and let $\delta(\varepsilon) > 0$ be such as in Lemma 2. Let further $t_1, t_2 \in G$ be such that $\sup_{s \in G} |y(t_1 + s) - y(t_2 + s)| \leqq c^{-1} \delta(\varepsilon)$. Then $|\exp(x(t_1 + s) - x(t_2 + s)) - e| = |\exp(-x(t_2 + s))(y(t_1 + s) - y(t_2 + s))| \leqq \delta(\varepsilon)$ for $s \in G$. Since the set $(x(t_1 + s) - x(t_2 + s);\ s \in G)$ is connected, it follows from Lemma 2 that there exists $y \in B_0$ such that $|x(t_1 + s) - x(t_2 + s) - y| \leqq \varepsilon$ for $s \in G$. From this and from the almost periodicity of the function y it follows that the function x satisfies the condition 2 of Theorem 1.

Let us show further that the function x is uniformly continuous. Let $\varepsilon > 0$ be given and let $\varepsilon_1 = \min(\varepsilon, 3^{-1} \lg 2)$. Let U be a neighborhood of $0 \in G$ such that $|x(t) - x(0)| \leqq \varepsilon_1$ for $t \in U$ and $|y(t + s) - y(s)| \leqq c^{-1} \delta(\varepsilon_1)$ for $t \in U$ and $s \in G$. By the above argument, for any fixed $t \in U$ there exists $y \in B_0$ such that $|x(t + s) - x(s) - y| \leqq \varepsilon_1$ for $s \in G$ and in particular for $s = 0$ $|x(t) - x(0) - y| \leqq \varepsilon_1 \leqq 3^{-1} \lg 2$. On the other hand, we have $|x(t) - x(0)| \leqq 3^{-1} \lg 2$ and so $y = 0$ by Lemma 2, i.e., the function x is uniformly continuous.

By Theorem 1 the function x has the form $x = x_1 + x_2$, where x_1 is additive and $x_2 \in AP(G, B)$. It suffices now to prove that $y_1 \in AP(G, B)$, where $y_1(t) = \exp(x_1(t))$ $(t \in G)$. This assertion follows immediately from the known theorems about almost periodic functions and from the relation: $\exp(x_1(t)) = \exp(-x_1(-t)) = y^{-1}(-t) \cdot \exp(x_2(-t))$ $(t \in G)$.

$2 \to 1$: This implication is clear.

At the end we give a standard application of the preceding theorem to differential equations:

Theorem 3. *Let B be a commutative Banach algebra with unity e, $a \in AP(R, B)$ and let x be a solution of the equation*

$$(1) \qquad x'(t) = a(t)\, x(t)\,, \quad x(0) = e\,.$$

Then the following conditions are equivalent:

1. $x \in AP(R, B)$.
2. $\int_0^t a(s)\, ds = t\, M(a) + b(t)$, *where* $b, c \in AP(R, B)$, $c(t) = \exp(t\, M(a))$ $(t \in R)$.

Proof. 1 → 2: The solution x of the equation (1) has the form: $x(t) = \exp(\int_0^t a(s) \,. \, ds)$ $(t \in R)$. Let us prove that there exists $d < \infty$ such that $|\exp(-\int_0^t a(s)\, ds)| \leqq d$ for $t \in R$. Since $\mathrm{cl}(R(x))$ is compact, it suffices to prove that y is regular for $y \in \mathrm{cl}(R(x))$. Let $y \in \mathrm{cl}(R(x))$ be given and let $T = (t_n;\, n \in N)$ be such a sequence that $y = \lim x(t_n)$. Let $(s_n;\, n \in N)$ be a subsequence of T such that $\lim |x_1 - p_{s_n} x|_\infty = 0$, $\lim |a_1 - p_{s_n} a|_\infty = 0$ for some $x_1, a_1 \in AP(R, B)$. It may be easily seen that the function x_1 is the solution of the equation $x_1'(t) = a_1(t)\, x_1(t)$, $x_1(0) = y$ and hence $x_1(t) = \exp(\int_0^t a_1(s)\, ds)\, y$ for $t \in R$. Because of $e \in \mathrm{cl}(R(x_1))$, the element y must be regular.

Theorem 2 implies that $\int_0^t a(s)\, ds = u(t) + b(t)$ $(t \in R)$, where u is additive and $b, c \in AP(R, B)$ $(c(t) = \exp(u(t)),\ t \in R)$. It suffices to prove that $u(t) = t\, M(a)$ $(t \in R)$. It is very well known that any additive (continuous) function on R into B has the form: $u(t) = tz$ $(t \in R)$ for some $z \in B$. Hence we have $a(t) = z + b'(t)$ and from this it follows immediately that $z = M(a)$ (because of $M(b') = 0$).

Appendix. Proof of Lemma 2. Let B be a commutative Banach algebra with unity e. First let us mention some known properties of the exponential function exp in B $(\exp(x) = e + \sum_{n=1}^{\infty} (n!)^{-1} x^n)$:

1. The function exp is continuously Fréchet differentiable and $\exp'(x)(y) = \exp(x)\, y$ for $x, y \in B$ (and hence $|\exp'(x)| = |\exp(x)|$);
2. $\exp(x + y) = \exp(x)\exp(y)$ $(x, y \in B)$;
3. $|\exp(x)| \leqq e^{|x|}$ $(x \in B)$;
4. $|\exp(x) - \exp(y)| \leqq e^{|y|}(e^{|x-y|} - 1)$ $(x, y \in B)$.

Let $B_0 = (x \in B;\, \exp(x) = e)$. B_0 is obviously a nonvoid additive subgroup of B. For $x, y \in B$ we set

$$f(x, y) = y + \exp(-y)(e - \exp(y)) - \exp(-y)(\exp(x) - \exp(y) - \exp(y)(x - y))\,.$$

Let us note that $x \in B_0$ iff $x = f(x, y)$ at least for one $y \in B$. If $x \in B_0$ then $x = f(x, y)$ for all $y \in B$.

For $r > 0$ we shall denote $K(x, r) = (y \in B; |x - y| \leqq r)$.

Let $0 \leqq d < 1$, $r > 0$ and let us prove that for $|\exp(x) - e| \leqq d$ and for $y, z \in \in K(x, r)$ the following estimates hold:

$$|f(y, x) - x| \leqq d(1 - d)^{-1} + (e^r - 1) r, \tag{2}$$

$$|f(y, x) - f(z, x)| \leqq (e^r - 1) |y - z|. \tag{3}$$

Indeed, let $|\exp(x) - e| \leqq d < 1$. Then $|\exp(-x)| = |(e - (e - \exp(x)))^{-1}| \leqq \leqq (1 - |\exp(x) - e|)^{-1}$. Further, it holds $\exp(-x)(\exp(y) - \exp(x) - \exp(x) \,.\, (y - x)) = \int_0^1 \exp(t(y - x) - e)\, dt\, (y - x)$ and from this we obtain $|f(y, x) - - x| \leqq |\exp(x) - e|(1 - |\exp(x) - e|)^{-1} + \int_0^1 (e^{t|y-x|} - 1)\, dt\, |y - x| \leqq \leqq d(1 - d)^{-1} + (e^r - 1) r$ for $|y - x| \leqq r$, which proves (2).

For $y, z \in B$ we have further $|f(y, x) - f(z, x)| = |g(y) - g(z)|$ where $g(y) = = \int_0^1 (\exp(t(y - x)) - e)\, dt\, (y - x)$. The function g is continuously Frechet differentiable and it holds $g'(y)(z) = \int_0^1 (\exp(t(y - x))(e + t(y - x)) - e)\, dt\, z$, which yields for $y \in K(x, r)$ the estimate $|g'(y)| \leqq \int_0^1 (e^{t|y-x|} - 1)\, dt + + \int_0^1 e^{t|y-x|} t|y - x|\, dt \leqq e^r - 1$. Hence for $y, z \in K(x, r)$ it is $|g(y) - g(z)| = = |\int_0^1 g'(z + t(y - z))\, dt\, (y - z)| \leqq (e^r - 1) |y - z|$, which proves (3).

Let $x \in B_0$ and $r \in (0, \lg 2)$. From the estimates (2) and (3) and from the Banach contraction theorem we obtain that the equation $y = f(y, x)$ has the unique solution in $K(x, r)$, namely x. From this we obtain that for $x, y \in B_0$ and $x \neq y$ it holds $|x - y| \geqq \lg 2$.

Let us denote $h(r) = r - (e^r - 1) r$. It is clear that for some $r_0 > 0$ it is $h(r) > 0$ for $r \in (0, r_0\rangle$ (obviously $r_0 < \lg 2$). Let $\varepsilon > 0$ be given and let $r \in (0, r_0\rangle$ be such that $r \leqq \min(\varepsilon, 3^{-1} \lg 2)$. Let us set $\delta(\varepsilon) = h(r)(1 + h(r))^{-1}$. Then $h(r) = \delta(\varepsilon) \,.\, (1 - \delta(\varepsilon))^{-1}$ or $r = \delta(\varepsilon)(1 - \delta(\varepsilon))^{-1} + (e^r - 1) r$ and also $e^r - 1 < 1$. Let $M \subset B$ be a connected set such that $|\exp(x) - e| \leqq \delta(\varepsilon)$ for $x \in M$. Then the estimates (2), (3) and the Banach contraction theorem imply that in $K(x, r)$ there exists a solution of the equation $y = f(y, x)$, i.e., to any $x \in M$ there exists $y_x \in B_0$ such that $|x - y_x| \leqq r \leqq \varepsilon$. Because of the facts that the set M is connected and $r \leqq 3^{-1} \lg 2$ it follows easily from the above that $y_x = y \in B_0$ for $x \in M$.

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