Vladimír Lovicar

Almost periodicity of solutions of the equation $x'(t) = A(t)x(t)$ with unbounded commuting operators

Časopis pro pěstování matematiky, Vol. 100 (1975), No. 1, 36--45

Persistent URL: `http://dml.cz/dmlcz/117866`

Časopis pro pěstování matematiky, roč. 100 (1975), Praha

ALMOST PERIODICITY OF SOLUTIONS OF THE EQUATION $x'(t) = A(t)\,x(t)$ WITH UNBOUNDED COMMUTING OPERATORS

VLADIMÍR LOVICAR, Praha

(Received September 24, 1973)

The aim of this paper is to obtain sufficient conditions for almost periodicity of all solutions of the equation

$$(1) \qquad x'(t) = A(t)\,x(t)\,, \quad x(0) = x_0\,,$$

in a Banach space B under the main assumption that $(A(t);\, t \in R)$ (in all paper R is the real line and $R^+ = (t \in R;\, t \geqq 0))$ is a commuting system of operators in B. Before obtaining this result (Theorem 2), general questions concerning solutions of the equation (1) and some elementary properties of commuting (unbounded) operators are considered.

All Banach spaces in this paper are supposed complex, semigroup (group) of operators in a Banach space always means strongly continuous semigroup (group) of operators. All notations are standard (see e.g. [1]), except the notions of ϱ-operator, S-operator and G-operator (Definitions 1 and 3), which are introduced only in order to simplify the formulation of assertions in this paper.

Let B be a Banach space and let A be a linear operator in B. We shall denote by $D(A)$ and $R(A)$ the domain (which is supposed non empty) and the range, respectively, of the operator A. If A is an injective operator, the operator A^{-1} may be defined by

$$D(A^{-1}) = R(A)\,,$$

$$A^{-1}x = y \quad \text{for} \quad x \in D(A^{-1})\,, \quad \text{where } y \text{ is such that} \quad x = Ay\,.$$

It is clear that $R(A^{-1}) = D(A)$. Further it may be easily seen that the following lemma holds:

Lemma 1. *Let A_1, A_2 be linear operators in a Banach space B such that*

1. $D(A_2) \supset R(A_1)$;
2. $R(A_2) \subset D(A_1)$;

3. $A_2A_1x = x$ *for* $x \in D(A_1)$;
4. $A_1A_2x = x$ *for* $x \in D(A_2)$.

Then A_1 *is an injective operator and* $A_1^{-1} = A_2$.

We shall denote by $\varrho(A)$ the set of regular points of the operator A, i.e., $\varrho(A) = (\lambda \in C;\ (\lambda E - A)^{-1}$ is bounded and everywhere defined) (here C denotes the complex plane and E the identity on B). For $\lambda \in \varrho(A)$ the operator $(\lambda E - A)^{-1}$ is denoted by $R(\lambda, A)$. It may be easily seen that $\varrho(A)$ is an open subset of C.

Definition 1. Let A be a linear operator in a Banach space B. We shall say that A is a *ϱ-operator* if $\varrho(A) \neq \emptyset$.

Lemma 2. *Let A be an injective ϱ-operator in a Banach space B. Then A^{-1} is also a ϱ-operator.*

Proof. Since $\varrho(A)$ is a nonempty open subset of C, there exists $\lambda \in \varrho(A)$, $\lambda \neq 0$. Let us prove that $\lambda^{-1} \in \varrho(A^{-1})$ and $R(\lambda^{-1}, A^{-1}) = -\lambda AR(\lambda, A) = \lambda E - \lambda^2 R(\lambda, A)$. Indeed, let S denote the operator $-\lambda AR(\lambda, A)$. Then the operator S is bounded and everywhere defined (hence $D(S) = B \supset R(\lambda^{-1}E - A^{-1})$) and $R(S) \subset R(A) = D(\lambda^{-1}E - A^{-1})$. Further, for $x \in D(\lambda^{-1}E - A^{-1})$ we have $S(\lambda^{-1}E - A^{-1})\,x = (\lambda E - \lambda^2 R(\lambda, A))\,\lambda^{-1}x + \lambda R(\lambda, A)\,x = x$. Finally, for $x \in D(S)$ it holds $(\lambda^{-1}E - A^{-1})\,Sx = (\lambda^{-1}E - A^{-1})(-\lambda A)\,R(\lambda, A)\,x = (-A + \lambda E)\,R(\lambda, A)\,x = x$. The assertion of the Lemma follows from Lemma 1.

Definition 2. Let A_1, A_2 be ϱ-operators in a Banach space B. We shall say that the operators A_1, A_2 *commute* (or that A_1 *commutes with* A_2) if for any $\lambda_1 \in \varrho(A_1)$, $\lambda_2 \in \varrho(A_2)$ it holds $R(\lambda_1, A_1)\,R(\lambda_2, A_2) = R(\lambda_2, A_2)\,R(\lambda_1, A_1)$. We shall say that $(A_j;\ j \in M)$ is a commuting system of operators if for any $j \in M$, A_j is a ϱ-operator and A_j commutes with A_k for $j, k \in M$.

Note. It may be easily seen that bounded everywhere defined operators A_1, A_2 commute in the sense of Definition 2 iff $A_1A_2 = A_2A_1$.

Lemma 3. *Let A_1, A_2 be ϱ-operators in a Banach space B. Let there exist $\lambda_1 \in \varrho(A_1)$ and $\lambda_2 \in \varrho(A_2)$ such that $R(\lambda_1, A_1)$ commutes with $R(\lambda_2, A_2)$. Then the operators A_1, A_2 commute.*

Proof. Let $\mu_1 \in \varrho(A_1)$. Then the operator $S = (\mu_1 E - A_1)\ R(\lambda_1, A_1) = (\mu_1 - \lambda_1)\,R(\lambda_1, A_1) + E$ is a bounded operator with bounded inverse $S^{-1} = (\lambda_1 E - A_1)\,R(\mu_1, A_1) = (\lambda_1 - \mu_1)\,R(\mu_1, A_1) + E$. The operator $R(\lambda_2, A_2)$ commutes with the operator S and hence $R(\lambda_2, A_2)$ commutes also with the operator S^{-1}. This implies immediately that the operator $R(\lambda_2, A_2)$ commutes with $R(\mu_1, A_1)$ for any $\mu_1 \in \varrho(A_1)$. By similar calculation it may be shown that $R(\mu_1, A_1)$ commutes with $R(\mu_2, A_2)$ for any $\mu_1 \in \varrho(A_1)$ and $\mu_2 \in \varrho(A_2)$.

Lemma 4. *Let A_1, A_2 be ϱ-operators in a Banach space B and let A_1 be injective. Then the following conditions are equivalent:*

1. *A_1 commutes with A_2.*
2. *A_1^{-1} commutes with A_2.*

Proof. The assertion follows immediately from the proof of Lemma 2 and from Lemma 3.

Lemma 5. *Let A_1, A_2 be ϱ-operators in a Banach space B. Then the following conditions are equivalent:*

1. *A_1 commutes with A_2*
2. *For any $\lambda \in \varrho(A_1)$ and $x \in D(A_2)$ it is $R(\lambda, A_1)\, x \in D(A_2)$ and $A_2R(\lambda, A_1)\, x = R(\lambda, A_1)\, A_2x$.*
3. *There exists $\lambda \in \varrho(A_1)$ such that for any $x \in D(A_2)$ it holds $R(\lambda, A_1)\, x \in D(A_2)$ and $A_2R(\lambda, A_1)\, x = R(\lambda, A_1)\, A_2x$.*

Proof. $1 \to 2$: Let $\mu \in \varrho(A_2)$ and $x \in D(A_2)$. Then $x = R(\mu, A_2)\, y$ for some $y \in B$ and hence for any $\lambda \in \varrho(A_1)$ it holds $R(\lambda, A_1)\, x = R(\lambda, A_1)\, R(\mu, A_2)\, y = R(\mu, A_2)\, R(\lambda, A_1)\, y$. From this we obtain $R(\lambda, A_1)\, x \in D(A_2)$ and further $A_2R(\lambda, A_1)\, x = A_2R(\mu, A_2)\, R(\lambda, A_1)\, y = (\mu R(\mu, A_2) - E)\, R(\lambda, A_1)\, y = R(\lambda, A_1) \,.\, (\mu R(\mu, A_2) - E)\, y = R(\lambda, A_1)\, A_2x$.

$3 \to 1$: Let $\mu \in \varrho(A_2)$ and $y \in B$. Then for $x = R(\mu, A_2)\, y \in D(A_2)$ we have $(\mu E - A_2)\, R(\lambda, A_1)\, x = R(\lambda, A_1)\, (\mu E - A_2)\, x$ and hence $(\mu E - A_2)\, R(\lambda, A_1) \,.\, R(\mu, A_2)\, y = R(\lambda, A_1)\, y$. From this it follows that $R(\lambda, A_1)\, R(\mu, A_2)\, y = R(\mu, A_2) \,.\, R(\lambda, A_1)\, y$ and A_1 commutes with A_2 by Lemma 3.

Lemma 6. *Let A_1, A_2 be ϱ-operators in a Banach space B and let A_2 be bounded (everywhere defined). Then the following conditions are equivalent:*

1. *A_1 commutes with A_2.*
2. *For any $x \in D(A_1)$ it is $A_2x \in D(A_1)$ and $A_1A_2x = A_2A_1x$.*

Proof. $1 \to 2$: It is clear that A_1 commutes with $\lambda E - A_2$ for any $\lambda \in C$. For sufficiently large λ, the operator $R(\lambda, A_2)$ exists, $0 \in \varrho(R(\lambda, A_2))$ and $R(0, R(\lambda, A_2)) = A_2 - \lambda E$. By Lemma 4, the operator A_1 commutes with $R(\lambda, A_2)$ and so by Lemma 5 for any $x \in D(A_1)$ it is $(A_2 - \lambda E)\, x \in D(A_1)$ and $A_1(A_2 - \lambda E)\, x = (A_2 - \lambda E)\, A_1x$, i.e. $A_2x \in D(A_1)$ and $A_1A_2x = A_2A_1x$.

$2 \to 1$: This implication follows immediately from the proof of $1 \to 2$, because all implications used in $1 \to 2$ are actually equivalences.

Lemma 7. *Let A be a ϱ-operator in a Banach space B. Further, let A_1 be a linear operator in B defined on $D(A)$ and such that for some $\lambda \in \varrho(A)$ the operator $A_1R(\lambda, A)$*

is a bounded operator and $A_1R(\lambda, A)x = R(\lambda, A)A_1x$ *for* $x \in D(A)$. *Then the operator* $A_1R(\mu, A)$ *is bounded for* $\mu \in \varrho(A)$ *and* $A_1R(\mu, A)x = R(\mu, A)A_1x$ *for* $\mu \in \varrho(A)$ *and* $x \in D(A)$. *Moreover, for sufficiently small* $\varepsilon > 0$ *the operator* $A + \varepsilon A_1$ *is a* ϱ*-operator which commutes with* A.

Proof. For $\mu \in \varrho(A)$ we have $A_1R(\mu, A) = A_1R(\lambda, A) + (\lambda - \mu)A_1R(\lambda, A) \cdot R(\mu, A)$ and hence $A_1R(\mu, A)$ is a bounded operator. Further we have for $x \in D(A)$ $A_1R(\mu, A)x = A_1R(\lambda, A)x + (\lambda - \mu)A_1R(\lambda, A)R(\mu, A)x$, $R(\mu, A)A_1x = R(\lambda, A) \cdot A_1x + (\lambda - \mu)R(\lambda, A)R(\mu, A)A_1x$ which implies $(E - (\lambda - \mu)R(\lambda, A)) \cdot (A_1R(\mu, A)x - R(\mu, A)A_1x) = 0$; the operator $E - (\lambda - \mu)R(\lambda, A) = (\mu E - A)R(\lambda, A)$ is injective and hence $A_1R(\mu, A)x = R(\mu, A)A_1x$ for $x \in D(A)$ and $\mu \in \varrho(A)$.

Let $\varepsilon < (|A_1R(\lambda, A)|)^{-1}$. Then the operator $(E - \varepsilon A_1R(\lambda, A))^{-1}$ exists and is bounded. Let us denote $S = R(\lambda, A)(E - \varepsilon A_1R(\lambda, A))^{-1}$. Then we have$D(S) \supset R(\lambda E - A - \varepsilon A_1)$, $R(S) \subset D(A)$, $(\lambda E - A - \varepsilon A_1)Sx = (\lambda E - A - \varepsilon A_1)R(\lambda, A) \cdot (E - \varepsilon A_1R(\lambda, A))^{-1}x = x$ for $x \in D(S)$ and $S(\lambda E - A - \varepsilon A_1)x = R(\lambda, A) \cdot (E - \varepsilon A_1R(\lambda, A))^{-1}(E - \varepsilon A_1R(\lambda, A))(\lambda E - A)x = x$ for $x \in D(A)$. This means by Lemma 1 that the operator $A + \varepsilon A_1$ is a ϱ-operator. Further we have $(\lambda E - A - \varepsilon A_1)(R(\lambda, A + \varepsilon A_1)R(\lambda, A) - R(\lambda, A)R(\lambda, A + \varepsilon A_1)) = R(\lambda, A) - R(\lambda, A + \varepsilon A_1) + \varepsilon R(\lambda, A)A_1R(\lambda, A + \varepsilon A_1) = R(\lambda, A) - R(\lambda, A + \varepsilon A_1) + R(\lambda, A) \cdot (-E + (\lambda E - A)R(\lambda, A + \varepsilon A_1)) = 0$ end hence the operator $A + \varepsilon A_1$ commutes with A.

Lemma 8. *Let* A *be a* ϱ*-operator in a Banach space* B. *Further let* A_1, A_2 *be linear operators in* B *defined on* $D(A)$ *such that for some* $\lambda_j \in \varrho(A)$ *the operators* $A_jR(\lambda_j, A)$ *are bounded and* $A_jR(\lambda_j, A)x = R(\lambda_j, A)A_jx$ *for* $x \in D(A)$ $(j = 1, 2)$. *Finally let he operators* $A_1R(\lambda_1, A)$, $A_2R(\lambda_2, A)$ *commute. Then for sufficiently small* $\varepsilon > 0$, $(A + \varepsilon A_1, A + \varepsilon A_2, A)$ *forms a commuting system of operators.*

Proof. It follows from Lemma 7 that for sufficiently small $\varepsilon > 0$ the operators $A + \varepsilon A_j$ are ϱ-operators which commute with A. It suffices to prove that the operators $A + \varepsilon A_1$, $A + \varepsilon A_2$ commute but this follows immediately from the relations $R(\lambda_1, A + \varepsilon A_1) = R(\lambda_1, A)(E - \varepsilon A_1R(\lambda_1, A))^{-1}$, $R(\lambda_2, A + \varepsilon A_2) = R(\lambda_2, A) \cdot (E - \varepsilon A_2R(\lambda_2, A))^{-1}$.

Definition 3. Let A be a linear operator in a Banach B. We shall say that the operator A is an *S-operator* (*G-operator*) if A generates a strongly continuous semigroup (group) of operators in B.

It is clear that any S-operator is a ϱ-operator. If A is an S-operator, we shall denote by $T(A) = (T(A, t);\ t \in R^+)$ the semigroup of operators generated by A. On the other hand, by $A(T)$ the generator of the semigroup of operators $T = (T(t);\ t \in R^+)$ is denoted.

Lemma 9. *Let A_1, A_2 be commuting ϱ-operators in a Banach space B. Moreover, let A_2 be an S-operator. Then the closure of the operator $A_1 \mid D(A_1) \cap D(A_2)$ equals A_1.*

Proof. We have to prove that for any $x \in D(A_1)$ there exists a sequence $(x_n;\ n \in N) \subset D(A_1) \cap D(A_2)$ such that $x = \lim x_n$ and $A_1 x = \lim A_1 x_n$. Let $(\lambda_n; n \in N)$ be a sequence of positive numbers such that $R(\lambda_n, A_2)$ exists for $n \in N$ and $\lambda_n \to +\infty$ for $n \to \infty$. It is very well known that $y = \lim \lambda_n R(\lambda_n, A_2)\, y$ for any $y \in B$. Now let $x \in D(A_1)$ be given. We obtain from Lemma 5 that for any $n \in N$ it is $x_n = \lambda_n R(\lambda_n, A_2)\, x \in D(A_1) \cap D(A_2)$, $A_1 x_n = \lambda_n R(\lambda_n, A_2)\, A_1 x$ and $x = \lim x_n$, $A_1 x = \lim A_1 x_n$ by the above.

Lemma 10. *Let A_1 be a ϱ-operator and let A_2 be an S-operator in a Banach space B. Then the following conditions are equivalent:*

1. *A_1 commutes with A_2.*
2. *$R(\lambda, A_1)$ commutes with $T(A_2, t)$ for any $\lambda \in \varrho(A_1)$ and $t \in R^+$.*
3. *There exists $\lambda \in \varrho(A_1)$ such that $R(\lambda, A_1)$ commutes with $T(A_2, t)$ for any $t \in R^+$.*

Proof. $1 \to 2$: Let $\lambda \in \varrho(A_1)$ and $t \in R^+$. Further let $(\mu_n;\ n \in N)$ be a sequence of positive numbers such that $R(\mu_n, A_2)$ exist for $n \in N$ and $\lim \mu_n = +\infty$. For any $x \in B$ we have $T(A_2, t)\, x = \lim_{n \in N} \exp\left(-t\mu_n(E - \mu_n R(\mu_n, A_2))\right) x$ and since the operators $R(\lambda, A_1)$, $\exp\left(-t\mu_n(E - \mu_n R(\mu_n, A_2))\right)$ commute, it holds $R(\lambda, A_1)\, T(A_2, t)\, x = \lim_{n \in N} \exp\left(-t\mu_n(E - \mu_n R(\mu_n, A_2))\right) R(\lambda, A_1)\, x = T(A_2, t)\, R(\lambda, A_1)\, x$ for $x \in B$.

$3 \to 1$: Let μ be a sufficiently large positive number. Then the operator $R(\mu, A_2)$ exists and $R(\mu, A_2)\, x = \int_0^\infty e^{-\mu t} T(A_2, t)\, x\, \mathrm{d}t$ for $x \in B$. Hence we have for any $x \in B$ $R(\lambda, A_1)\, R(\mu, A_2)\, x = \int_0^\infty e^{-\mu t} R(\lambda, A_1)\, T(A_2, t)\, x\, \mathrm{d}t = \int_0^\infty e^{-\mu t} T(A_2, t)\, R(\lambda, A_1)\, x dt = R(\mu, A_2)\, R(\lambda, A_1)\, x$ and the operators A_1, A_2 commute by Lemma 3.

Definition 4. Let T_1, T_2 be semigroups of operators in a Banach space B. We shall say that T_1, T_2 *commute* (or that T_1 *commutes with* T_2) if for any $t, s \in R^+$, $T_1(t)$ commutes with $T_2(s)$.

Lemma 11. *Let A_1, A_2 be S-operators in a Banach space B. Then the following conditions are equivalent:*

1. *A_1 commutes with A_2.*
2. *$T(A_1)$ commutes with $T(A_2)$.*

Proof. $1 \to 2$: The operator $R(\lambda, A_1)$ commutes with $T(A_2, t)$ ($\lambda \in \varrho(A_1)$, $t \in R^+$) by Lemma 10. If we repeat once more the procedure from the proof of implication $1 \to 2$ of Lemma 10, we obtain that the operators $T(A_1, t)$, $T(A_2, s)$ commute for $t, s \in R^+$.

$2 \to 1$: Let λ be a sufficiently large positive number. Then $\lambda \in \varrho(A_1)$ and for any $x \in B$ it is $R(\lambda, A_1)\, x = \int_0^\infty e^{-\lambda s} T(A_1, s)\, x \,\mathrm{d}s$. This implies for $t \in R^+$ and $x \in B$ $R(\lambda, A_1)\, T(A_2, t)\, x = \int_0^\infty e^{-\lambda s} T(A_1, s)\, T(A_2, t)\, x \,\mathrm{d}s = \int_0^\infty e^{-\lambda s} T(A_2, t)\, T(A_1, s)\, x \,\mathrm{d}s = T(A_2, t)\, R(\lambda, A_1)\, x$ and hence A_1 commutes with A_2 by Lemma 10.

Lemma 12. *Let A_1, A_2 be commuting S-operators in a Banach space B. Further let us set $T(t) = T(A_1, t)\, T(A_2, t)$ $(t \in R^+)$. Then $T = (T(t);\ t \in R^+)$ is a semigroup of operators in B and $A(T)$ equals the closure of the operator $A_1 + A_2$.*

Proof. It is clear that T is a semigroup of operators in B. Further we have $A(T)\, x = \lim_{t \to 0+} t^{-1}(T(t)\, x - x) = \lim_{t \to 0+} (T(A_2, t)\,(t^{-1}(T(A_1, t)\, x - x)) + t^{-1}(T(A_2, t)\, x - x)) = A_1 x + A_2 x$ for $x \in D(A_1) \cap D(A_2)$. Hence it is sufficient to show that for some $\lambda \in \varrho(A(T))$ the set $(\lambda E - A_1 - A_2)\,(D(A_1) \cap D(A_2))$ is dense in B. Since T, $T(A_1)$, $T(A_2)$ is a commuting system of semigroups of operators, we have $R(\lambda, A(T))\,.\,R(\mu, A_1)\, R(\mu, A_2) = R(\mu, A_1)\, R(\mu, A_2)\, R(\lambda, A(T))$ $(\lambda \in \varrho(A(T)),\ \mu \in \varrho(A_1) \cap \varrho(A_2))$ by Lemma 11. This implies that for any $x \in B$ it is $y = \mu^2 R(\lambda, A(T))\, R(\mu, A_1)\,.\,R(\mu, A_2)\, x \in D(A_1) \cap D(A_2)$ and $(\lambda E - A_1 - A_2)\, y = \mu^2 R(\mu, A_1)\, R(\mu, A_2)\, x$. Since $\lim_{\mu \to +\infty} \mu^2 R(\mu, A_1)\, R(\mu, A_2)\, x = x$ for any $x \in B$, the set $(\lambda E - A_1 - A_2)\,(D(A_1) \cap D(A_2))$ is dense in B.

Lemma 13. (Lumer G. - Phillips R. S.) *Let A be a closed linear operator in a Banach space B,* $\mathrm{cl}\,(D(A)) = B$. *Further let J be the duality mapping on B, i.e.* $Jx = (x^* \in B^*;\ (x, x^*) = |x|^2,\ |x| = |x^*|)$. *Then the following conditions are equivalent*:

1. *The operator A is a G-operator for which* $|T(A, t)| \leqq e^{\omega|t|}$ *for* $t \in R$.
2. $|\mathrm{Re}\,(Ax, Jx)| \leqq \omega |x|^2$ *for* $x \in D(A)$ *and there exist* $\lambda_1, \lambda_2 \in \varrho(A)$ *such that* $\mathrm{Re}\, \lambda_1 < -\omega < +\omega < \mathrm{Re}\, \lambda_2$.

Proof. This Lemma is an easy modification of Lumer-Phillips Theorem (see e.g. [1]).

Lemma 14. *Let A be an S-operator in a Banach space B and let* $s \in R^+$. *Then the operator sA is an S-operator and it holds* $T(sA, t) = T(A, st)$ *for* $t \in R^+$. *If A is a G-operator, then sA is a G-operator for* $s \in R$ *and* $T(sA, t) = T(A, st)$ *for* $t \in R$.

Lemma 15. *Let A be a G-operator in a Banach space B for which* $|T(A, t)| \leqq e^{\omega|t|}$ *for* $t \in R$. *Further, let* A_1 *be a linear operator in B defined on* $D(A)$ *and such that* $|\mathrm{Re}\,(A_1 x, Jx)| \leqq \omega_1 |x|^2$ *for* $x \in D(A)$. *Finally let there exist* $\lambda \in \varrho(A)$ *such that* $A_1 R(\lambda, A)$ *is a bounded operator and* $A_1 R(\lambda, A)\, x = R(\lambda, A)\, A_1 x$ *for* $x \in D(A)$. *Then the closure* $\bar{A}_1$ *of the operator* A_1 *is a G-operator and* $T(\bar{A}_1)$ *commutes with* $T(A)$.

Proof. Let $\operatorname{Re}\lambda_1 < -\omega < \omega < \operatorname{Re}\lambda_2$. Then we have by Lemma 7 that for sufficiently small $\varepsilon > 0$ $\lambda_1, \lambda_2 \in \varrho(A + \varepsilon A_1)$ $(j = 1, 2)$ and hence $A + \varepsilon A_1$ is a G-operator by Lemma 13. Further, $A + \varepsilon A_1$ commutes with A by Lemma 7 and the assertion of the Lemma follows immediately from Lemma 12 and Lemma 14.

Lemma 16. *Let linear operators A, A_j $(j = 1, 2)$ fulfil the same condition as the operators A, A_1 in Lemma 15. Then the following conditions are equivalent:*

1. *$T(\bar{A}_1)$ commutes with $T(\bar{A}_2)$.*
2. *$A_1R(\lambda, A)$ commutes with $A_2R(\lambda, A)$ at least for one $\lambda \in \varrho(A)$.*

Proof. $1 \to 2$: It follows easily from Lemmas 7,11 and 12 that for sufficiently small $\varepsilon > 0$, $(A, A + \varepsilon A_1, A + \varepsilon A_2)$ forms a commuting system of operators and $M = \varrho(A) \cap \varrho(A + \varepsilon A_1) \cap \varrho(A + \varepsilon A_2) \neq \emptyset$. Let $\lambda \in M$. Then by Lemma 5 for any $x \in D(A)$ it holds $AR(\lambda, A + \varepsilon A_1)\, x = R(\lambda, A + \varepsilon A_1)\, Ax$ and this easily implies that the operators $(\lambda E - A)\, R(\lambda, A + \varepsilon A_1)$, $(\lambda E - A)\, R(\lambda, A + \varepsilon A_2)$ commute. Hence also the operators $((\lambda E - A)\, R(\lambda, A + \varepsilon A_1))^{-1} = E - \varepsilon A_1 R(\lambda, A)$ and $((\lambda E - A)\, R(\lambda, A + \varepsilon A_2))^{-1} = E - \varepsilon A_2 R(\lambda, A)$ commute.

$2 \to 1$: This implication follows easily from Lemma 8 and 12.

Lemma 17. *Let A be a G-operator in a Banach space B for which $|T(A, t)| \leqq e^{\omega|t|}$ for $t \in R$. Further let M be a directed set and let $(A(j); j \in M)$ be a system of linear operators in B defined on $D(A)$ with the following properties:*

1. *$|\operatorname{Re}(A(j)\, x, Jx)| \leqq \omega_1 |x|^2$ for $x \in D(A)$;*
2. *There exists $\lambda \in \varrho(A)$ such that $A(j)\, R(\lambda, A)$ is a bounded operator for $j \in M$;*
3. *$A(j)\, R(\lambda, A)\, x = R(\lambda, A)\, A(j)\, x$ for $j \in M$ and $x \in D(A)$;*
4. *$A(j)\, R(\lambda, A)$ commutes with $A(k)\, R(\lambda, A)$ for $j, k \in M$;*
5. *For any $x \in D(A)$ there exists $\lim_{j \in M} A(j)\, x = A_1 x$.*

Then the closure $\bar{A}_1$ of the operator A_1 is a G-operator, $(\bar{A}_1, \overline{A(j)}\ (j \in M), A)$ forms a commuting system of operators and $T(\bar{A}_1, t)\, x = \lim_{j \in M} T(\overline{A(j)}, t)\, x$ for $x \in B$ and $t \in R$.

Proof. It follows from Lemma 16 that the operator $\overline{A(j)}$ $(j \in M)$ is a G-operator for which $|T(\overline{A(j)}, t)| \leqq e^{\omega_1|t|}$ for $t \in R$ and $(A, \overline{A(j)}\ (j \in M))$ forms a commuting system of operators.

Let us prove that the operators A, A_1 fulfil the conditions of Lemma 15. For any $x \in D(A)$ and $x^* \in Jx$ it is $|\operatorname{Re}(A_1 x, x^*)| = \lim_{j \in M} |\operatorname{Re}(A(j)\, x, x^*)| \leqq \omega_1 |x|^2$. Further it holds $A_1 R(\lambda, A)\, x = \lim_{j \in M} A(j)\, R(\lambda, A)\, x$ for any $x \in B$ and hence the operator $A_1 R(\lambda, A)$ is bounded. Since $A(j)\, R(\lambda, A)\, x = R(\lambda, A)\, A(j)\, x$ for any $j \in M$ and $x \in D(A)$, we have $A_1 R(\lambda, A)\, x = R(\lambda, A)\, A_1 x$ for $x \in D(A)$. Hence, by Lemma 15, the closure $\bar{A}_1$ of the operator A_1 is a G-operator and $\bar{A}_1$ commutes with A.

Since $A(j)\, R(\lambda, A)$ commutes with $A(k)\, R(\lambda, A)$ for $j, k \in M$, we easily obtain that $A_1 R(\lambda, A)$ commutes with $A(j)\, R(\lambda, A)$ for $j \in M$ and hence, by Lemma 16, $\bar{A}_1$ commutes with $\overline{A(j)}$ for $j \in M$.

We have further for $x \in D(A)$ $|T(\bar{A}_1, t)\, x - T(\overline{A(j)}, t)\, x| = |T(\overline{A(j)}, t) \,.$ $. (T(\overline{A_1 - A(j)}, t)\, x - x)| \leqq e^{\omega_1|t|} |\int_0^t T(\overline{A_1 - A(j)}, s)\, (A_1 - A(j))\, x \,\mathrm{d}s| \leqq k(\omega_1, t) \,.$ $. |A_1 x - A(j)\, x|$ and hence $T(\bar{A}_1, t)\, x = \lim_{j \in M} T(\overline{A(j)}, t)\, x$ for $x \in D(A)$. Since the operators $T(A(j), t)$ are bounded uniformly with respect to $j \in M$, we obtain that $T(\bar{A}_1, t)\, x = \lim_{j \in M} T(\overline{A(j)}, t)\, x$ for $x \in B$.

Theorem 1. *Let A, $A(t)$ $(t \in R)$ be a commuting system of G-operators in a Banach space B for which $|T(A, t)| \leqq e^{\omega|t|}$, $|T(A(s), t)| \leqq e^{\omega_1|t|}$ for $t, s \in R$. Further let $D(A(t)) \supset D(A)$ for $t \in R$ and let the function $t \to A(t)\, x$ be continuous for $x \in D(A)$. Let us define for $t, s \in R$ the operator $C(t, s)$ by: $C(t, s)\, x = \int_s^t A(r)\, x \,\mathrm{d}r$ $(x \in D(A))$. Then the closure $\int_s^t A(r)\,\mathrm{d}r$ of the operator $C(t, s)$ is a G-operator for $t, s \in R$. Let us denote further*

$$\mathcal{T}(t, s) = T\left(\int_s^t A(r)\,\mathrm{d}r, 1\right) \quad (t, s \in R)$$

Then it holds:

1. *$\mathcal{T}(t, s)$ $(t, s \in R)$ is a commuting system of operators.*
2. *$|\mathcal{T}(t, s)| \leqq e^{\omega_1|t-s|}$ for $t, s \in R$.*
3. *The function $(t, s) \to \mathcal{T}(t, s)\, x$ is continuous on $R \times R$ for any $x \in B$.*
4. *$\mathcal{T}(t, t) = E$, $\mathcal{T}(t, s)\, \mathcal{T}(s, r) = \mathcal{T}(t, r)$ for $t, s, r \in R$.*
5. *The function $t \to \mathcal{T}(t, s)\, x$ is continuously differentiable for $x \in D(A)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{T}(t, s)\, x = A(t)\, \mathcal{T}(t, s)\, x \quad (t \in R)\,.$$

Proof. First, let us prove that $A(t)\, R(\lambda, A)$ is a bounded operator for $t \in R$ and $\lambda \in \varrho(A)$. Since the operator $A(t)\, R(\lambda, A)$ is everywhere defined, it suffices to prove that it is closed. Let $x_n \to x$, $A(t)\, R(\lambda, A)\, x_n \to y$. Then $y_n = R(\lambda, A)\, x_n \in D(A(t))$, $y_n \to R(\lambda, A)\, x$ and $A(t)\, y_n \to y$ and hence $A(t)\, R(\lambda, A)\, x = y$ since the operator $A(t)$ is closed.

The set $(A(t)\, R(\lambda, A)\, x;\ t \in \langle a, b\rangle)$ is bounded for any compact set $\langle a, b\rangle \subset R$ and hence there exists a constant $k(a, b) < +\infty$ such that $|A(t)\, R(\lambda, A)| \leqq k(a, b)$ for $t \in \langle a, b\rangle$. This easily implies that $C(t, s)\, R(\lambda, A)$ is a bounded operator for $t, s \in R$. It is also clear that $C(t, s)\, R(\lambda, A)\, x = R(\lambda, A)\, C(t, s)\, x$ for $x \in D(A)$. It follows from Lemma 15 that the closure $\int_s^t A(r)\,\mathrm{d}r$ of the operator $C(t, s)$ is a G-operator for $t, s \in R$.

It is easy to see that by Lemma 16 A, $A(t)$ $(t \in R)$, $\int_s^t A(r)\,\mathrm{d}r$ $(t, s \in R)$ form a commuting system of operators which implies the assertion 1 of the Lemma. Lemma 13 implies the assertion 2 of the Theorem. The assertion 3 follows from Lemma 17. The assertion 4 follows easily from the relation $C(t, s) + C(s, r) = C(t, r)$ $(t, s, r \in R)$. Hence it suffices to prove the validity of the assertion 5.

First let us notice that if A_1, A_2 are commuting G-operators then for $t \in R$ and $x \in D(A_1)$ it is $T(A_2, t)\, x \in D(A_1)$ and $A_1 T(A_2, t)\, x = T(A_2, t)\, A_1 x$ (Lemma 10).

Let $x \in D(A)$. Then we have $\lim_{h \to 0} h^{-1}(\mathscr{T}(t + h, s)\, x - \mathscr{T}(t, s)\, x) = \lim_{h \to 0} h^{-1}(T(\int_s^{t+h} A(r)\, dr, 1)\, x - T(\int_s^t A(r)\, dr, 1)\, x) = \mathscr{T}(t, s) \lim_{h \to 0} h^{-1}(T(\int_t^{t+h} A(r) \,.\, dr, 1)\, x - x) = \mathscr{T}(t, s) \lim_{h \to 0} h^{-1} \int_0^1 T(\int_t^{t+h} A(r)\, dr, u) (\int_t^{t+h} A(r)\, x\, dr)\, du = \mathscr{T}(t, s) \,.\, A(t)\, x = A(t)\, \mathscr{T}(t, s)\, x$ since $|\int_0^1 T(\int_t^{t+h} A(r)\, dr, u)\, (h^{-1} \int_t^{t+h} A(r)\, x\, dr)\, du - A(t)\, x| \leqq |\int_0^1 (T(\int_t^{t+h} A(r)\, dr, u) - E)\, A(t)\, x\, du| + |\int_0^1 T(\int_t^{t+h} A(r)\, dr, u) \,.\, (h^{-1} \int_t^{t+h} A(r)\, x\, dr - A(t)\, x)\, du| \to 0$ for $h \to 0$.

Definition 5. Let the operators A, $A(t)$, $\mathscr{T}(t, s)$ $(t, s \in R)$ in a Banach space B be the same as in Theorem 1. Then the function $t \to \mathscr{T}(t, s)\, x$ is said to be a solution of the equation

$$x'(t) = A(t)\, x(t)\,, \quad x(s) = x\,. \tag{2}$$

Lemma 18. *Let the operators A, $A(t)$ $(t \in R)$ in a Banach space B fulfil the assumptions of Theorem 1 and let moreover the function $t \to A(t)\, x$ be almost periodic for any $x \in D(A)$. Let A_0 be a linear operator in B defined on $D(A)$ by*

$$A_0 x = \lim_{t \to \infty} t^{-1} \int_0^t A(s)\, x\, ds\,, \quad x \in D(A)\,.$$

Then the closure $\bar{A}_0$ of the operator A_0 is a G-operator and $(A, A(t)$ $(t \in R)$, $\int_s^t A(r)\, dr$ $(t, s \in R)$, $\bar{A}_0)$ form a commuting system of operators.

Proof. This Lemma is another easy application of Lemmas 16 and 17.

Theorem 2. *Let A, $A(t)$ $(t \in R)$ be a commuting system of G-operators in a Banach space B for which $|T(A, t)| \leqq e^{\omega|t|}$, $|T(A(s), t)| \leqq e^{|\omega_1||t|}$ for $t, s \in R$. Further let $D(A(t)) \supset D(A)$ for $t \in R$ and let the function $t \to A(t)\, x$ be almost periodic for any $x \in D(A)$. Let us define the operators A_0, $C(t)$ $(t \in R)$ on $D(A)$ by*

$$A_0 x = \lim_{t \to \infty} t^{-1} \int_0^t A(s)\, x\, ds\,, \quad C(t)\, x = \int_0^t A(s)\, x\, ds - t A_0 x\,.$$

Finally let us suppose that the following assumptions are fulfilled:

1. *The function $t \to T(\bar{A}_0, t)\, x$ is almost periodic for any $x \in B$;*
2. $|\mathrm{Re}\, (C(t)\, x, Jx)| \leqq \omega_2 |x|^2$ *for some $\omega_2 > 0$ and for $t \in R$, $x \in D(A)$;*
3. *the function $t \to C(t)\, x$ is almost periodic for any $x \in D(A)$.*

Then all solutions of the equation (2) are almost periodic.

Proof. It is clear that it suffices to prove that the function $t \to \mathscr{T}(t, 0)\, x$ is almost periodic for $x \in B$. For any $t \in R$ the operator $\mathscr{T}(t, 0)$ has the form $\mathscr{T}(t, 0) = T(\bar{A}_0, t)\, T(\overline{C(t)}, 1)$. The function $t \to T(\bar{A}_0, t)\, x$ is almost periodic for $x \in B$ which implies that there exists a constant $k_1 < \infty$ such that $|T(\bar{A}_0, t)| \leqq k_1$ for $t \in R$. If we prove that the function $t \to T(\overline{C(t)}, 1)\, x$ is almost periodic for $x \in B$, then the almost periodicity of the function $t \to \mathscr{T}(t, 0)\, x$ for $x \in B$ will follow from the well known theorem about the almost periodicity of composed function. Since the operators $T(\overline{C(t)}, 1)$ are uniformly bounded $(|T(\overline{C(t)}, 1)| \leqq e^{\omega_2}$ for $t \in R)$, it suffices to prove that the function $t \to T(\overline{C(t)}, 1)\, x$ is almost periodic for $x \in D(A)$.

Let $x \in D(A)$. Then we have for $t, s \in R$ $T(\overline{C(t)}, 1)\, x - T(\overline{C(s)}, 1)\, x = T(\overline{C(s)}, 1)\, . \, (T(\overline{C(t) - C(s)}, 1)\, x - x) = T(\overline{C(s)}, 1)\, (\int_0^1 T(\overline{C(t) - C(s)}, r)\, (C(t)\, x - C(s)\, x)\, \mathrm{d}r)$. This implies that there exists a constant $k_2 = k_2(\omega_2)$ such that $|T(\overline{C(t)}, 1)\, x - T(\overline{C(s)}, 1)\, x| \leqq k_2 |C(t)\, x - C(s)\, x|$. The almost periodicity of the function $t \to T(C(t), 1)\, x$ follows from the assumption 3 of the Theorem.

Note. Let $\bar{A}_0$ be such a G-operator in a weakly complete Banach space B that $T(\bar{A}_0)$ is a bounded group of operators, i.e. $|T(\bar{A}_0, t)| \leqq k < \infty$ for $t \in R$. Moreover, let $\sigma(\bar{A}_0) = C \setminus \varrho(\bar{A}_0)$ be at most countable. Then it follows from Theorem 1 of [3] and from Theorem 3 of [4] that the assumption 3 of Theorem 2 is fulfilled, i.e., the function $t \to T(\bar{A}_0, t)\, x$ is almost periodic for any $x \in B$.

Bibliography

[1] *Yosida K.:* Functional analysis (Springer 1965).

[2] *Amerio L., Prouse G.:* Almost periodic functions and functional equations (Van Nostrand, New York 1971).

[3] *Любич Ю. И.:* Об условиях полноты собственных векторов корректного оператора (Успехи мат. наук *XVII* (1963), 165—171).

[4] *Lovicar V.:* Weakly almost periodic solutions of linear equations in Banach spaces (Čas. pěst. mat. *98* (1973), pp 126—129).

Author's address: 115 67 Praha 1, Žitná 25 (Matematický ústav ČSAV).