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A GENERALIZATION OF THE TORSION FORM

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The well-known torsion form of a linear connection on an n-dimensional manifold $M$ is the exterior covariant derivative of the canonical $\mathbb{R}^n$-valued form $\varphi$ of the bundle of linear frames. As a natural generalization of $\varphi$, we have introduced the canonical $(\mathbb{R}^n \oplus \mathfrak{g})$-valued form $\theta$ of the first prolongation $W^1(P)$ of an arbitrary principal fibre bundle $P(B, G)$. $n = \dim B$, [5]. In a similar way, we define the torsion form of a connection on $W^1(P)$ to be the exterior covariant derivative of $\theta$. This concept generalizes also the torsion form of a linear connection of higher order in the sense of YUEN, [11]. Using a result by ŠVEC, we find the structure equations of $\theta$. We also deduce that the connections on $W^1(P)$ are in a one-to-one correspondence with certain reductions of the second semi-holonomic prolongation $\overline{W}^2(P)$ of $P$, and a connection on $W^1(P)$ is without torsion if and only if the corresponding reduction is holonomic. In the special case of a linear connection, these results were established by KOBAYASHI, [3], and LIBERMANN, [8]. In conclusion, we treat the prolongation $p(\Gamma, \Lambda)$ of a connection $\Gamma$ on $P$ with respect to a linear connection $\Lambda$ on the base manifold, [7], and we find a necessary and sufficient geometric condition for $p(\Gamma, \Lambda)$ to be without torsion. — Standard terminology and notation of the theory of jets are used throughout the paper, see, e.g., [10]. Our investigations are carried out in the category $C^\infty$.

1. Let $G$ and $H$ be two Lie groups. Assume that every $g \in G$ determines an automorphism $\bar{g} : H \to H$ such that the mapping $g \mapsto \bar{g}$ is a right action of $G$ on $H$. Consider the corresponding semi-direct product $G \ltimes H$, i.e., the multiplication in $G \ltimes H$ is given by

\[(g_1, h_1)(g_2, h_2) = (g_1g_2, \bar{g}_2(h_1)h_2), \quad g_1, g_2 \in G, \quad h_1, h_2 \in H.\]

We have natural injections $G \to G \ltimes H$, $g \mapsto (g, e_H)$ and $H \to G \ltimes H$, $h \mapsto (e_G, h)$. In this sense, Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of $G$ and $H$ form two complementary subspaces of the Lie algebra of $G \ltimes H$. One verifies directly that

\[\text{ad} (g, e_H)(e_G, h) = (e_G, \bar{g}^{-1}(h)).\]
Consider further a principal fibre bundle $P(B, G, n)$. Introduce a projection $P \times H \rightarrow B$, $(u, h) \mapsto \pi(u), u \in P$, $h \in H$, and define a right action of $G \times H$ on $P \times H$ by

$$(u, h_1)(g, h_2) = (ug, g(h_1)h_2).$$

**Lemma 1.** $P \times H$ with action (3) is a principal fibre bundle $(P \times H)(B, G \times H)$.

**Proof** is straightforward.

Obviously, $P \approx P \times \{e_H\}$ is a reduction of $P \times H$ to the subgroup $G \subset G \times H$. In view of (2), we can apply the result by Švec, [9], p. 572. This proves

**Lemma 2.** Let $\omega = \omega_1 \oplus \omega_2$ be a $(g \oplus h)$-valued connection form on $P \times H$ and $\bar{\omega}_1$ or $\bar{\omega}_2$ the restriction of $\omega_1$ or $\omega_2$ to $P$, respectively. Then $\bar{\omega}_1$ is a connection form and $\bar{\omega}_2$ is an $h$-valued tensorial form of type $ad\ G$. Conversely, if $\bar{\omega}$ is a connection form on $P$ and $\varphi$ is an $h$-valued tensorial form of type $ad\ G$ on $P$, then there is a unique connection form on $P \times H$ such that its restriction to $P$ is $\bar{\omega} \oplus \varphi$.

In particular, let $g$ be a representation of $G$ on a finite dimensional vector space $V$. For $A \in \mathfrak{g}$, $A = j^1_0(\gamma(t))$ and $B \in V$, we set

$$A \cdot B = \lim_{t \to 0} \frac{1}{t} [\varphi(\gamma(t))(B) - B].$$

This defines a bilinear map $\mathfrak{g} \times V \rightarrow V$, $(A, B) \mapsto A \cdot B$. Since $V$ is an Abelian group and $g \mapsto \varphi(g^{-1})$ is a right action of $G$ on $V$, we can construct the semi-direct product $G \times V$. Let $\omega$ be a connection form on $P$ and $\varphi$ a $V$-valued tensorial 1-form of type $\rho$ on $P$. We have the situation of Lemma 2 and one verifies easily that formula (2.23) of [9] is equivalent to the following

**Proposition 1.** It is

$$d\varphi = -\omega \cdot \varphi + D\varphi,$$

where $D\varphi$ is the covariant exterior derivative of $\varphi$ with respect to $\omega$ and $\omega \cdot \varphi$ means the 2-form on $P$ defined by the extension of bilinear map (4).

2. Consider now the first prolongation $W^1(P)$ of a principal fibre bundle $P(B, G)$, [5]. We recall that $W^1(P) = H^1(B) \oplus J^1P$ is a principal fibre bundle over $B$ with structure group $G^1_n = L^1_n \rtimes T^1_n(G)$ ( = the semi-direct product with respect to the action $S \mapsto SY$ of $L^1_n$ on $T^1_n(G)$, $Y \in L^1_n$, $S \in T^1_n(G)$), $n = \dim B$. There are two canonical principal fibre bundle homomorphisms $\beta : W^1(P) \rightarrow P$ and $\lambda : W^1(P) \rightarrow H^1(B)$. In [5], we have introduced the canonical $(\mathbb{R}^n \oplus \mathfrak{g})$-valued form $\theta$ of $W^1(P)$ and we have deduced that $\theta$ is a pseudotensorial form of type $\rho$, where the representation $\rho$ of $G^1_n$ on $\mathbb{R}^n \oplus \mathfrak{g}$ is defined by formula (12) of [5]. If $\Gamma$ is a connection on $W^1(P)$, then the covariant absolute derivative $D\theta$ of $\theta$ will be called the torsion of $\Gamma$.
Remark 1. If we consider the trivial one-element group $G = \{e\}$ and the trivial bundle $B \times \{e\}$, then $W^1(B \times \{e\}) = H^1(B)$ and $\theta$ coincides with the canonical $\mathbb{R}^n$-valued form of $H^1(B)$. Hence we get really a generalization of the linear case.

Remark 2. Using the identification $\tilde{H}^r(B) \approx W^1(\tilde{H}^{r-1}(B))$ of [5], we obtain the inclusion $\tilde{H}^r(M) \subset W^1(\tilde{H}^{r-1}(M))$. Further, the restriction of the canonical form of $W^1(\tilde{H}^{r-1}(M))$ to $\tilde{H}^r(M)$ is the canonical form of $\tilde{H}^r(M)$. In this interpretation, our results generalize the investigation of the torsion form of a higher order linear connection by Yuen, [11].

Proposition 2. (Structure equations of $\theta$.) Let $\omega$ be a connection form on $W^1(P)$. Then we have

\[ d\theta = -\omega \cdot \theta + \frac{1}{2} [\omega_1, \omega_1] + D\theta, \]

where the $\mathbb{R}$-valued form $[\omega_1, \omega_1]$ is considered an $(\mathbb{R}^n \otimes g)$-valued form with zero component in $\mathbb{R}^n$.

Proof is based on Proposition 1. However, $\theta$ is not horizontal. That is why we shall first consider the tensorial form $\tilde{\theta} = \theta h$, i.e. $\tilde{\theta}(X) = \theta(hX)$, where $hX$ means the horizontal component of the vector $X \in T(W^1(P))$. By Proposition 1,

\[ d\tilde{\theta} = -\langle \omega_2, \tilde{\theta}_0 \rangle + D\tilde{\theta}_0, \]

\[ d\tilde{\theta}_1 = -[\omega_1, \tilde{\theta}_1] - \langle \omega_3, \tilde{\theta}_0 \rangle_g + D\tilde{\theta}_1. \]

Further, let $Y$ be a vertical vector on $W^1(P)$, which is the value of the fundamental vector field determined by an element $A \in g_1^n$. By the definition of $\theta$, [5], we have $\theta(Y) = A_1$. Hence $\theta = \tilde{\theta} + \omega_1$, where the g-valued form $\omega_1$ is considered an $(\mathbb{R}^n \otimes g)$-valued form with zero component in $\mathbb{R}^n$. Substituting it into (8), we obtain

\[ d\theta = -\langle \omega_2, \theta_0 \rangle + D\theta_0, \]

\[ d\theta_1 = -[\omega_1, \theta_1] - \langle \omega_3, \theta_0 \rangle_g + [\omega_1, \omega_1] + d\omega_1 - D\omega_1 + D\theta_1. \]
According to the structure equations of $\omega$, it is

\begin{equation}
\omega_1 = -\frac{1}{2}([\omega_1, \omega_1] + D\omega_1).
\end{equation}

Comparing (9) and (10), we deduce (7), QED.

3. We have remarked in [4] that a connection on a principal fibre bundle $P(B, G)$ can be defined as a $G$-invariant cross section $P \to J^1P$. Consider a connection $\Gamma$ on $W^1(P)$ in such a form, i.e. $\Gamma : W^1(P) \to J^1 W^1(P)$. We have $J^1 W^1(P) = J^1 (H^1(B) \oplus J^1 P) = J^1 H^1(B) \oplus J^2 P$. There is a standard identification $\chi : J^1 H^1(B) \approx H^2(B)$ sending an element $Z = j^1_x \varphi \in J^1 H^1(B)$, $\varphi(x) = j^1_0 \psi(y)$ into $\chi(Z) = j^1_0[\varphi(\psi(y)) t^{-1}_y] \in H^2(B)$, where $t_y : \mathbb{R}^n \to \mathbb{R}^n$ is the translation $z \mapsto z + y$.

On the other hand, the second semi-holonomic prolongation $W^2(P)$ of $P$ is equal to $H^2(B) \oplus J^2 P$, [5], so that the jet inclusion $J^2 P \subset J^2 P$ induces the inclusion $W^2(P) \subset J^1 W^1(P)$. We define the reduction $R(\Gamma) \subset W^2(P)$ determined by a connection $\Gamma : W^1(P) \to J^1 W^1(P)$ to be the intersection

\begin{equation}
R(\Gamma) = \Gamma(W^1(P)) \cap W^2(P).
\end{equation}

Consider the induced connection $\Gamma_0 = \beta_* \Gamma : P \to J^1 P$. We recall, [6], that $R(\Gamma_0) := H^1(B) \oplus \Gamma_0(P)$ is a reduction of $W^1(P)$ to the subgroup $L^1_n \times i_2(G) \subset G^{1}_n$, where $i_2 : G \to T^2_1(G)$ is the canonical injection $g \mapsto j^1_0 \vartheta$, $\vartheta$ being the constant mapping $x \mapsto g$, $x \in \mathbb{R}^n$.

**Lemma 3.** We have $\Gamma(u) \in R(\Gamma)$ if and only if $u \in R(\Gamma_0) \subset W^1(P)$.

**Proof.** Let $\Gamma(u) = j^1_x \varphi$, where $\varphi = (\varphi_1, \varphi_2)$ is a local cross section of $H^1(B) \oplus J^1 P$. The condition for $j^1_x \varphi_2$ to be semi-holonomic is $\varphi_2(x) = j^1_0 (j^1_x \varphi_2) = j^1_0 (\beta \varphi) = \Gamma_0(j^0_0 u)$, where $j^0_0 : J^1 P \to P$ is the jet projection. This is equivalent to $u \in R(\Gamma_0)$, QED.

Consider further the canonical injections $i_2 : G \to T^2_1(G)$, $g \mapsto j^1_0 \vartheta$ and $i : L^1_n \to \to L^2_n$. The last mapping can be geometrically described as follows. If $Y \in L^1_n$, $Y = j^1_0 \psi(y)$, then

\begin{equation}
i(Y) = j^1_0[t_{\psi(0)} Y^{-1}].
\end{equation}

Our next assertion generalizes a result by Libermann, [8].

**Proposition 3.** $R(\Gamma)$ is a reduction of $W^2(P)$ to the subgroup $i(L^1_n) \times i_2(G) \subset L^2_n \times T^2_1(G) = G^2_n$. Conversely, every reduction $Q$ of $W^2(P)$ to $i(L^1_n) \times i_2(G)$ determines a unique connection $\Gamma(Q)$ on $W^1(P)$ such that $Q = R(\Gamma(Q))$.

**Proof.** Put $\Gamma(v, \Gamma_0(u)) = (Z, T) \in J^1 H^1(B) \oplus J^2 P$, $u \in P$, $v \in H^1(B)$, and $Z =$
Starting from the fact that $\Gamma$ is $G^*_n$-invariant and using the formula for the action of $G^*_n$ on $W^1(P)$, we find

\[(13) \quad i^*(v, \Gamma_0(ug)) = (\varphi(y) Y, \sigma(y) \cdot (i_1(g) Y^{-1} \varphi^{-1}(y)))\]

$Y \in L^1_n$, $g \in G$. On the other hand, the formula for the action of $G^*_2$ on $W^2(P)$ yields

\[(14) \quad i^*(\sigma(z), T)(i^*(y), \sigma(y) \cdot (i_2(g) i^*(Y) \varphi^{-1}((y)))) = \varphi(z) Y, \sigma(y) \cdot (i_2(g) i^*(Y) \varphi^{-1}((y)))\]

see [5]. The relation $\varphi(z) i^*(y) = (i_2(g) i^*(Y) \varphi^{-1}((y)))$ is known from the linear case, [8]. Further, the injection $i_2 : G \to T^2_n(G)$ can be also expressed as $g \mapsto j_0^1[i_1(g) t^{-1}].$ Then we find easily $i_2(g) i^*(Y) \varphi^{-1}((y)) = j_2^1(i_1(g) Y^{-1} \varphi^{-1}((y)))$. Comparing with (13), we conclude that $R(\Gamma)$ is a reduction to the subgroup $i(L^1_n) \times i_2(G)$. The converse assertion can be proved quite similarly, QED.

We shall also need another geometric characterization of $R(\Gamma)$. We recall that a semi-holonomic connection of the second order on $P$ is a $G$-invariant cross section $P \to J^2P$, [4]. For every $(v, u) \in H^1(B) \oplus P$, define $\mu(\Gamma)(v, u) = p_2(\Gamma(v, \Gamma_0(u)))$, where $p_2 : J^1 W^1(P) \to J^2P$ is the product projection. According to Lemma 3, the values of $\mu(\Gamma)$ lie in $J^2P$.

**Lemma 4.** For every $Y \in L^1_n$, it is $\mu(\Gamma)(v, u) = \mu(\Gamma)(vY, u)$.

**Proof.** Let $\Gamma(v, \Gamma_0(u)) = j_2^1(\varphi_1(y), \varphi_2(y))$. Since $\Gamma$ is invariant, we have $\Gamma(vY, \Gamma_0(u)) = j_2^1(\varphi_1(Y) Y, \varphi_2(y))$. QED.

Thus, we may consider $\mu(\Gamma)$ to be a cross section $P \to J^2P$.

**Proposition 4.** $\mu(\Gamma) : P \to J^2P$ is a semi-holonomic connection of the second order on $P$.

**Proof.** We have to prove that $\mu(\Gamma)$ is $G$-invariant. But this is a simple consequence of Proposition 3, QED.

Denote by $A = \lambda_\bullet \Gamma$ the induced connection on $H^1(B)$ and by $R(A)$ the corresponding reduction of $\bar{H}^2(B)$. Our previous consideration implies

\[(14) \quad R(\Gamma) = R(A) \oplus \mu(\Gamma)(P) .\]

4. The following assertion generalizes a result by Kobayashi, [3].

**Proposition 5.** It is $R(\Gamma) \subset W^2(P)$ if and only if $D\theta = 0$.

**Proof.** We first deduce a lemma. Since $\bar{W}^2(P) \subset W^1(W^1(P))$, every $U \in \bar{W}^2(P)$ determines a mapping $\bar{U}^{-1} : T_u(W^1(P)) \to R^* \oplus g^1_\nu$, where $u \in W^1(P)$ is the underlying jet of $U$, [5]. Denote by $q : W^1(P) \to B$ the bundle projection.

**Lemma 5.** Let $M$ be a submanifold of $W^1(P)$ such that $q | M$ is a submersion.
Let $\sigma : M \rightarrow W^2(P)$ be a cross section and $\theta$ the $(\mathbb{R}^n \oplus g_4^1)$-valued form on $M$ constructed by means of $\sigma$, i.e. $\theta \mid T_u(M) = \sigma(u)^{-1} \mid T_u(M)$, $u \in M$. Then $\sigma(M) \subset W^2(P)$ if and only if

$$d\theta_0 = -\langle \theta_2, \theta_0 \rangle, \quad d\theta_1 = -\frac{1}{2}[\theta_1, \theta_1] - \langle \theta_3, \theta_0 \rangle_G.$$

Proof of Lemma 5. For $M = W^1(P)$, the assertion was deduced by direct evaluation by Dekrét, [1]. Using the coordinates of [5] or [1], we have local coordinates $a_{ij}$ on fibred manifold $W^2(P) \rightarrow W^1(P)$, $i, j, \ldots = 1, \ldots, n$, $\lambda = 1, \ldots, n + \dim G$. The subspace $W^2(P) \subset W^2(P)$ is characterized by $a_{ij} = a_{ij}$. Consider (locally) a cross section $\sigma : W^1(P) \rightarrow W^2(P)$ extending $\sigma$. Let $\sigma$ be given by some functions $f^i_{ij}$, so that $\sigma$ is given by $f^i_{ij} = f^i_{ij} \mid M$. Denote by $\theta$ the $(\mathbb{R}^n \oplus g_4^1)$-valued form on $W^1(P)$ constructed by means of $\sigma_1$. The evaluations by Dekrét imply (in coordinates)

$$d\bar{\theta}^i = \bar{\theta}^i \wedge \bar{\theta}^j + f^i_{jk} \bar{\theta}^j \wedge \bar{\theta}^k,$$

$$d\bar{\theta}^\alpha = -\frac{1}{2}e^{\alpha}_{\beta\gamma} \bar{\theta}^\beta \wedge \bar{\theta}^\gamma + \bar{\theta}^i \wedge \bar{\theta}^i + f^\alpha_{ijk} \bar{\theta}^j \wedge \bar{\theta}^k,$$

$\alpha = n + 1, \ldots, n + \dim G$. Restricting (16) to $M$, we find that (15) holds if and only if $f^i_{ij} = f^i_{ij}$, thus proving Lemma 5.

We are now in position to prove Proposition 5. Denote by $\tilde{\omega} = \tilde{\omega}_1 \oplus \tilde{\omega}_2 \oplus \tilde{\omega}_3$ or $\bar{\theta} = \bar{\theta}_0 \oplus \bar{\theta}_1$ the restriction of the connection form or the canonical form $\theta$ to $R(\Gamma)$, respectively. By the definition of $R(\Gamma)$ and by Lemma 3, it is $\tilde{\omega}_1 = \bar{\theta}_1$ and $\tilde{\omega}_2 \oplus \tilde{\omega}_3$ is the $(\mathbb{R}^n \oplus g_4^1)$-valued form constructed by means of the cross section $\Gamma \mid R(\Gamma)$, according to (7), we have

$$d\bar{\theta}_0 = -\langle \tilde{\omega}_2, \bar{\theta}_0 \rangle + D\bar{\theta}_0,$$

$$d\bar{\theta}_1 = -\frac{1}{2}[\bar{\theta}_1, \bar{\theta}_1] - \langle \tilde{\omega}_3, \bar{\theta}_0 \rangle_G + D\bar{\theta}_1.$$

Then Proposition 5 follows from Lemma 5, QED.

5. In particular, if $\Gamma$ is a connection on $P$ and $\Lambda$ is a linear connection on $B$, then the prolongation $p(\Gamma, \Lambda)$ of $\Gamma$ with respect to $\Lambda$ is an interesting special connection on $W^1(P)$, [7].

Proposition 6. Connection $p(\Gamma, \Lambda)$ is without torsion if and only if $\Gamma$ is integrable and $\Lambda$ is without torsion.

Proof. As a direct consequence of the definition, we have $\mu(p(\Gamma, \Lambda)) = \Gamma'$, where $\Gamma'$ means the prolongation of $\Gamma$ in the sense of Ehresmann, [2]. According to (14), it is $R(p(\Gamma, \Lambda)) = R(\Lambda) \oplus \Gamma'(P)$. By a result by Kobayashi, [3] (or as a special case of Proposition 5), $R(\Lambda) \subset H^2(B)$ if and only if $\Lambda$ is without torsion. On the other hand, according to Ehresmann, [2], $\Gamma'(P) \subset J^2P$ if and only if $\Gamma$ is integrable. By Proposition 5 we prove our assertion, QED.
References


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