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ON A CERTAIN DISTANCE BETWEEN ISOMORPHISM CLASSES OF GRAPHS

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In [1] V. G. VIZING advanced the problem to find a criterion for recognizing whether two given graphs with n vertices each are isomorphic to subgraphs of the same graph with $n + 1$ vertices. Here we shall study this problem more generally.

Theorem 1. *Let n be a positive integer, k a non-negative integer. Let G_1 and G_2 be two graphs, each with n vertices. Then the following two assertions are equivalent:*

(1) *There exists a graph G with at most $n + k$ vertices having two induced subgraphs G'_1 and G'_2 such that $G'_1 \cong G_1$, $G'_2 \cong G_2$.*

(2) *There exist isomorphic graphs G''_1, G''_2, each with at least $n - k$ vertices, such that G''_1 is an induced subgraph of G_1 and G''_2 is an induced subgraph of G_2.*

Remark. The graphs mentioned in this theorem may be directed or undirected, with or without loops and multiple edges.

Proof. (1) $\Rightarrow$ (2). If two sets with n elements each are subsets of a set with at most $n + k$ elements, then their intersection has evidently at least $n - k$ elements. Thus the intersection of vertex sets of G'_1 and G'_2 has at least $n - k$ elements; this set induces a subgraph G'' of G and of both G'_1 and G'_2 and is isomorphic to a subgraph G''_1 of G_1 and to a subgraph G''_2 of G_2.

(2) $\Rightarrow$ (1). For the sake of simplicity assume G_1 and G_2 vertex-disjoint. (This is no substantial restriction, because we deal with isomorphism.) Let φ be an isomorphic mapping of G''_1 onto G''_2. We identify each vertex v of G''_1 with its image $\varphi(v)$. The graph obtained from G_1 and G_2 in this way will be denoted by G; evidently it has at most $n + k$ vertices. If we put $G'_1 = G_1$, $G'_2 = G_2$, the proof is complete.

On the system $\mathcal{S}_n$ of all isomorphism classes of undirected graphs with n vertices without loops and multiple edges we can introduce a distance δ defined so that if $\mathfrak{G}_1 \in \mathcal{S}_n$, $\mathfrak{G}_2 \in \mathcal{S}_n$ and $n + k$ is the least possible number of vertices of a graph containing induced subgraphs from the classes $\mathfrak{G}_1$ and $\mathfrak{G}_2$, then $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = k$. (An isomorphism class of graphs is the class of all graphs isomorphic to a given graph.)

Theorem 2. *Let $\mathcal{S}_n$ be the system of all isomorphism classes of undirected graphs with n vertices without loops and multiple edges. If $\mathfrak{G}_1 \in \mathcal{S}_n$, $\mathfrak{G}_2 \in \mathcal{S}_n$ and $n + k$ is the least possible number of vertices of a graph containing induced subgraphs from the classes $\mathfrak{G}_1$ and $\mathfrak{G}_2$, then denote $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = k$. The system $\mathcal{S}_n$ with the functional δ is a metric space.*

Proof. If $\mathfrak{G}_1 = \mathfrak{G}_2$, then the least possible number of vertices of a graph containing subgraphs from $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is n, because such a graph is an arbitrary graph from $\mathfrak{G}_1 = \mathfrak{G}_2$ and it has n vertices. Thus $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = 0$. Now if $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = 0$, there exists a graph with n vertices containing induced subgraphs from $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Each graph from a class of $\mathcal{S}_n$ has n vertices and a graph with n vertices contains exactly one induced subgraph with n vertices, namely itself. Therefore this graph belongs to both $\mathfrak{G}_1$ and $\mathfrak{G}_2$ and thus $\mathfrak{G}_1 = \mathfrak{G}_2$. The equality $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = \delta(\mathfrak{G}_2, \mathfrak{G}_1)$ for any $\mathfrak{G}_1, \mathfrak{G}_2$ follows from the definition of δ. Now let $\mathfrak{G}_1 \in S_n$, $\mathfrak{G}_2 \in S_n$, $\mathfrak{G}_3 \in S_n$ and let $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = k_{12}$, $\delta(\mathfrak{G}_2, \mathfrak{G}_3) = k_{23}$. There exists a graph G_{12} with $n + k_{12}$ vertices containing induced subgraphs $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$ and a graph G_{23} with $n + k_{23}$ vertices containing induced subgraphs $H_2 \in \mathfrak{G}_2$, $H_3 \in \mathfrak{G}_3$. As both G_2 and H_2 belong to $\mathfrak{G}_2$, we have $G_2 \cong H_2$ and there exists an isomorphic mapping ψ of G_2 onto H_2. This is an isomorphic mapping from G_{12} into G_{23}. By identifying u with $\psi(u)$ for each vertex u of G_2 from G_{12} and G_{23} we obtain a graph G. This graph has $(n + k_{12}) + (n + k_{23}) - n = n + k_{12} + k_{23}$ vertices and contains $G_1 \in \mathfrak{G}_1$ and $H_3 \in \mathfrak{G}_3$ as induced subgraphs. This means

$$\delta(\mathfrak{G}_1, \mathfrak{G}_3) \leqq k_{12} + k_{23} = \delta(\mathfrak{G}_1, \mathfrak{G}_2) + \delta(\mathfrak{G}_2, \mathfrak{G}_3),$$

which is the triangle inequality for δ.

Theorem 3. *Let $\mathfrak{G}_1, \mathfrak{G}_2$ be two isomorphism classes from $\mathcal{S}_n$. ($\mathcal{S}_n$ and δ have the same meaning as in Theorem 2.) Let $\overline{\mathfrak{G}}_1$ or $\overline{\mathfrak{G}}_2$ be the isomorphism class consisting of complements to the graphs of $\mathfrak{G}_1$ or $\mathfrak{G}_2$, respectively. Then*

$$\delta(\overline{\mathfrak{G}}_1, \overline{\mathfrak{G}}_2) = \delta(\mathfrak{G}_1, \mathfrak{G}_2).$$

Proof. There exists a graph G with $n + \delta(\mathfrak{G}_1, \mathfrak{G}_2)$ vertices containing induced subgraphs $G_1 \in \mathfrak{G}_1$, $G_2 \in \mathfrak{G}_2$. Then the complement $\bar{G}$ of G contains induced subgraphs $\bar{G}_1, \bar{G}_2$ which are complements of G_1, G_2 respectively. It has also $n + \delta(\mathfrak{G}_1, \mathfrak{G}_2)$ vertices and $\bar{G}_1 \in \overline{\mathfrak{G}}_1$, $\bar{G}_2 \in \overline{\mathfrak{G}}_2$, thus $\delta(\overline{\mathfrak{G}}_1, \overline{\mathfrak{G}}_2) \leqq \delta(\mathfrak{G}_1, \mathfrak{G}_2)$. On the other hand, interchanging $\mathfrak{G}_1, \overline{\mathfrak{G}}_1$ and $\mathfrak{G}_2, \overline{\mathfrak{G}}_2$ in our argument we obtain $\delta(\mathfrak{G}_1, \mathfrak{G}_2) \leqq \delta(\overline{\mathfrak{G}}_1, \overline{\mathfrak{G}}_2)$ and thus $\delta(\overline{\mathfrak{G}}_1, \overline{\mathfrak{G}}_2) = \delta(\mathfrak{G}_1, \mathfrak{G}_2)$.

Consider a graph $\mathcal{D}_n$ whose vertex set is $\mathcal{S}_n$ and in which two vertices $\mathfrak{G}_1 \in \mathcal{S}_n$, $\mathfrak{G}_2 \in \mathcal{S}_n$ are adjacent if and only if $\delta(\mathfrak{G}_1, \mathfrak{G}_2) = 1$. It is easy to prove that the distance of vertices in $\mathcal{D}_n$ is δ. The diameter of $\mathcal{D}_n$ is $n - 1$; this is the distance between the isomorphism class consisting of complete graphs and the isomorphism class consisting of graphs without edges. The distance between any other pair of vertices is less than $n - 1$, because if a graph G_1 has edges and is not complete, it contains both kinds of two-vertex subgraphs and thus there exists its two-vertex subgraph which is isomorphic to a subgraph of an arbitrary other graph G_2. (All considered graphs have n vertices.) According to Theorem 1 then there exists a graph with at most $2n - 2$ vertices containing induced subgraphs isomorphic to G_1 and G_2.

We have restricted our consideration to undirected graphs without loops and multiple edges. Nevertheless, Theorem 1 and Theorem 2 remain valid even if we consider undirected graphs with loops and multiple edges or directed graphs. In Theorem 3 we must consider graphs without loops and multiple edges; otherwise we could not speak about complements. If instead of $\mathcal{S}_n$ we consider the set of all isomorphism classes of graphs which may contain loops, then for the diameter of the corresponding graph we obtain the value n instead of $n - 1$; this is the distance between the isomorphism class consisting of some graphs with n vertices without loops and the isomorphism class consisting of some graphs with n vertices with a loop at each vertex.

The investigation of $\mathcal{D}_n$ seems to be very difficult, because for greater values of n it is difficult even to determine its vertex set. According to Theorem 3 we can assert that there exists an automorphism of $\mathcal{D}_n$ which maps each isomorphism class $\mathfrak{G}$ onto the isomorphism class $\overline{\mathfrak{G}}$ consisting of the complements of graphs from $\mathfrak{G}$. It would be interesting to find the radius of $\mathcal{D}_n$.

Reference

[1] *В. Г. Визинг:* Некоторые нерещенные задачи в теории графов. Успехи мат. наук *23* (1968), 117—134.

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