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ON THE LATTICE GROUP VALUED MEASURES

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In the paper we study some properties of non-negative lattice group valued measures on topological spaces. Naturally enough, this group is assumed to satisfy a certain regularity condition. Therefore, the first part is devoted to this condition, a generalization of the Alexandroff theorem being proved here. The second part is concerned with the product of measures and the third one with the Kolmogoroff consistency theorem.

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Let $G$ be an Abelian lattice ordered group, i.e. and Abelian group which is a lattice and which satisfies the implication: $x < y \Rightarrow x + z < y + z$. A group valued submeasure $\mu$ is a mapping $\mu : \mathcal{R} \to G$, where $\mathcal{R}$ is a ring of subsets of a space $X$, non-decreasing, subadditive, $\mu(\emptyset) = 0$ and upper semicontinuous in $\emptyset$ (i.e. $A_n \searrow \emptyset \Rightarrow \mu(A_n) \searrow 0$). An additive submeasure is called measure. (Of course, every measure is $\sigma$-additive.)

Definition 1. An Abelian lattice ordered group $G$ is weakly regular*) if it satisfies the following condition: Let $a \in G$, $a > 0$ and let $a_n \searrow O$ ($i \to \infty$), then there are such $i_1, i_2, \ldots$ that

$$a \leq \sum_{j=1}^{n} a_{i_j}$$

for no $n$.

As an example of a weakly regular group let us take the additive group $R$ of all real numbers. In this case it suffices to choose $i_k$ such that

$$a_k < \frac{a}{2^k}.$$  

*) We say "weakly" since there is a stronger notion of regularity used in [5].
Then

\[ a \leq \sum_{j=1}^{i} a_j \]

for some \( n \) implies

\[ a \leq \sum_{j=1}^{n} \frac{a}{2^j} < a , \]

which is impossible.

Now let us present two less trivial examples.

**Example 1.** Every linearly ordered group is weakly regular. First we construct the sequence \( \{i_j\}_{j=1}^{\infty} \). Since \( a_i \searrow O \) \((i \rightarrow \infty)\), it is also \( 2a_i = a_i + a_i \searrow O \), hence there is \( i_1 \) such that \( 2a_i < a \). Similarly there is \( i_2 \) such that \( 4a_i^2 < a \) and generally there is \( i_k \) such that \( 2^k a_i^k < a \). If \( a \leq \sum_{j=1}^{n} a_j^j \) then

\[
2^a a \leq 2^a a_{i_1} + 2^a a_{i_2}^2 + \ldots + 2^a a_{i_n}^n = \\
= 2^{a-1} 2^a a_{i_1} + 2^{a-2} 2^2 a_{i_2}^2 + \ldots + 1 \cdot 2^a a_{i_n}^n < \\
< 2^{a-1} a + 2^{a-2} a + \ldots + 1 \cdot a = (2^n - 1) a ,
\]

which is impossible.

**Example 2.** Every regular \( K \)-space is a weakly regular group. A regular \( K \)-space \((\text{see [6] Th. VI.5.2})\) is a linear semiordered space (= Riesz space = \( K \)-lineal) which is relatively complete and such that every sequence of convergent sequences has a common regulator of convergence. If \( b_n \searrow O \), then \( u > O \) is a regulator of convergence of \( \{b_n\}_{n=1}^{\infty} \) iff to any number \( \varepsilon > 0 \) there is \( n_0 \) such that \( b_n < \varepsilon u \) for every \( n \geq n_0 \). Hence \( b_n \searrow O \) is false iff to any \( u > O \) there is \( \varepsilon > 0 \) such that for any \( n_0 \) there is \( n \geq n_0 \) such that \( b_n < \varepsilon u \) is false. Now let \( a_n^i \searrow O \) \((i \rightarrow \infty, \ n = 1, 2, \ldots)\) and let \( u \) be the common regulator of convergence of all \( \{a_n^i\}_{i=1}^{\infty}, \ n = 1, 2, \ldots \). Given \( \varepsilon > 0 \) there is \( i_n \) such that

\[ a_n^i < \frac{\varepsilon}{2^n} u . \]

If

\[ a = \sum_{j=1}^{n_0} a_j^j < \sum_{j=1}^{n_0} \frac{\varepsilon}{2^j} u < \varepsilon u \]

then \( a < \varepsilon u \) for every \( \varepsilon < 0 \) which is a contradiction since \( a > O \).

In the paper we shall consider only regular measures.

**Definition 2.** Let \( \mathcal{C} \) be a family of subsets of a set \( X \). We say that \( \mathcal{C} \) is a **compact family** if \( \mathcal{C} \) is closed under finite intersections and every decreasing sequence of non-empty sets of \( \mathcal{C} \) has a non-empty intersection.
Definition 3. Let $\mathcal{R}$ be a ring of subsets of a set $X$, $\mathcal{C} \subset \mathcal{R}$, $\mathcal{C}$ a compact family. Let $\mu : \mathcal{R} \to G$ be a lattice group valued submeasure. We say that $\mu$ is inner regular if to any $E \in \mathcal{R}$ there are such sets $C_n \in \mathcal{C}$ ($n = 1, 2, \ldots$) that $C_n \subset C_{n+1} \subset E$ ($n = 1, 2, \ldots$) and
\[ \mu(E - C_n) \searrow 0. \]

The following theorem is a generalization of the Alexandroff theorem. Various other generalizations in the real-valued case are found in [4].

Theorem 1. Let $G$ be weakly regular, $\sigma$-complete,*) Abelian lattice-ordered group. Let $X$ be a topological space, $\mathcal{R}$ a ring of subsets of $X$. Let $\mu : \mathcal{R} \to G$, $\mu(\emptyset) = O$ be monotone, subadditive and inner regular. Then $\mu$ is upper semicontinuous in $\emptyset$.

Proof. Let $A_n \searrow \emptyset$ (i.e. $A_n \supset A_{n+1}$ ($n = 1, 2, \ldots$) and $\bigcap_{n=1}^{\infty} A_n = \emptyset$). We want to prove that $\mu(A_n) \searrow O$. Let us prove it indirectly. Since $G$ is $\sigma$-complete, there is $a > O$ such that $\mu(A_n) \geq a$. Since $\mu$ is inner regular, to any $n$ there are $C^l_n \in \mathcal{C}$ ($i = 1, 2, \ldots$) such that
\[ C^l_n \subset C^l_n+1 \subset A_n \quad (i = 1, 2, \ldots) \]
and
\[ \mu(A_n - C^l_n) \searrow O \quad (i \to \infty). \]

Put
\[ a_n = \mu(A_n), \quad a^l_n = \mu(A_n - C^l_n) \]
and choose $i_1, i_2, \ldots$ by Definition 1. Now put
\[ D_1 = C^l_1, \quad D_2 = C^l_2 \cap C^l_1, \ldots, \quad D_n = C^l_n \cap C^{l_{n-1}} \cap \ldots \cap C^l_1, \ldots. \]

Then $D_n \in \mathcal{C}$, $D_n \supset D_{n+1}$ ($n = 1, 2, \ldots$). We prove that $D_n \neq \emptyset$ ($n = 1, 2, \ldots$):

If $D_n = \emptyset$, then
\[ a \leq \mu(A_n) \leq \mu\left( \bigcup_{j=1}^{n} (A_j - C^l_j) \right) \cup \left( \bigcap_{j=1}^{n} C^l_j \right) \leq \]
\[ \leq \sum_{j=1}^{n} \mu(A_j - C^l_j) + \mu(D_n) = \sum_{j=1}^{n} \mu(A_j - C^l_j) = \sum_{j=1}^{n} a^l_j \]
which is impossible.

Since $D_n \supset D_{n+1}$, $D_n \in \mathcal{C}$, $D_n \neq \emptyset$ ($n = 1, 2, \ldots$), we have $\bigcap_{n=1}^{\infty} D_n \neq \emptyset$. But $D_n \subset A_n$ ($n = 1, 2, \ldots$), hence also $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$, which is a contradiction.

*) I.e. every bounded countable set has the supremum.
Now we want to prove a theorem on the product of two measures. Usually the product of two measures $\mu, \nu$ is defined as such a measure $\lambda$ in the cartesian product that

$$\lambda(E \times F) = \mu(E) \nu(F)$$

for all $E, F$ from the corresponding domains. However, in our general group $G$ we need not have any product. Hence we shall assume that there are given three groups $G_1, G_2, G$ and a mapping

$$\pi : G_1 \times G_2 \to G$$

satisfying some conditions. We shall need the following three simple conditions:

1. $\pi(a + b, c) = \pi(a, c) + \pi(b, c)$, $\pi(a', b' + c') = \pi(a', b') + \pi(a', c')$ for all $a, b, a' \in G_1$, $c, b', c' \in G_2$.

2. If $a \not\in O$, $b \not\in O$, $a \in G_1$, $b \in G_2$, then $\pi(a, b) \not\in O$.

3. If $a_n \not\in O$, $b_n \not\in O$, $a_n \in G_1$, $b_n \in G_2$ $(n = 1, 2, \ldots)$ then $\pi(a_n, b_n) \not\in O$.

**Theorem 2.** Let $\mathcal{R}_1$ or $\mathcal{R}_2$ be rings of subsets of $X_1$ or $X_2$ respectively. Let $\mu : \mathcal{R}_1 \to G_1$, $\nu : \mathcal{R}_2 \to G_2$ be inner regular measures. Let $G$ be weakly regular, $\sigma$-complete, Abelian, lattice-ordered group. Then there is exactly one $G$-valued measure $\lambda$ defined on the ring $\mathcal{R}$ generated by the family $\mathcal{D} = \{E \times F; E \in \mathcal{R}_1$, $F \in \mathcal{R}_2\}$ and such that

$$\lambda(E \times F) = \pi(\mu(E), \nu(F))$$

for all $E \in \mathcal{R}_1$, $F \in \mathcal{R}_2$.

**Proof.** Define first $\lambda_0 : \mathcal{D} \to G$ by the formula $\lambda_0(E \times F) = \pi(\mu(E), \nu(F))$. Evidently $\lambda_0$ is additive, monotone, $\lambda_0(\emptyset) = O$. Hence we can extend $\lambda_0$ to a function $\lambda : \mathcal{R} \to G$ by the formula

$$\lambda(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \lambda_0(A_i)$$

where $A_i$ are disjoint sets from $\mathcal{D} (i = 1, \ldots, n)$. The function $\lambda$ is also additive, nonnegative (and therefore monotone and subadditive). It suffices to prove that $\lambda$ is upper semicontinuous in $\emptyset$.

Let $\mathcal{C}_1, \mathcal{C}_2$ be compact families of subsets of $X_1$ or $X_2$ respectively. Let $\mathcal{C}$ consist of all finite unions of sets of the form $C \times D$ where $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$. Then $\mathcal{C}$ is a compact family.

Now let $A \in \mathcal{R}$. Then $A = \bigcup_{i=1}^m A_i = \bigcup_{i=1}^m (E_i \times F_i)$, where $A_i$ are pairwise disjoint.
Since $E_i \in \mathcal{R}_1$ and $\mu$ is inner regular, there are $C^n_i \subseteq C^{n+1}_i \subseteq E_i$ $(n = 1, 2, \ldots)$ such that

$$C^n_i \subseteq C^{n+1}_i \subseteq E_i \quad (n = 1, 2, \ldots)$$

and

$$\mu(E_i - C^n_i) \searrow O \quad (n \to \infty).$$

Similarly there are $D^n_i \subseteq \mathcal{C}_2$ $(n = 1, 2, \ldots)$ such that

$$D^n_i \subseteq D^{n+1}_i \subseteq F_i \quad (n = 1, 2, \ldots)$$

and

$$\nu(F_i - D^n_i) \searrow O \quad (n \to \infty).$$

By the third property of $\pi$ we have

$$\pi(A_i - C^n_i \times D^n_i) = \pi((E_i \times F_i) - (C^n_i \times D^n_i)) \leq \pi(\mu(E_i - C^n_i), \nu(F_i)) + \pi(\mu(E_i), \nu(F_i - D^n_i)) \searrow O \quad (n \to \infty).$$

Put $K_n = \bigcup (C^n_i \times D^n_i) (n = 1, 2, \ldots)$. Then $K_n \subseteq \mathcal{C}_2$, $K_n \subseteq K_{n+1} \subseteq A$ $(n = 1, 2, \ldots)$ and

$$\lambda(A - K_n) = \lambda(\bigcup_{i=1}^{m} A_i - \bigcup_{i=1}^{m} (C^n_i \times D^n_i)) =$$

$$= \lambda(\bigcup_{i=1}^{m} (A_i - C^n_i \times D^n_i)) = \sum_{i=1}^{m} \lambda(A_i - C^n_i \times D^n_i) \searrow O,$$

if $n \to \infty$. Hence $\lambda$ is regular and the proof is complete.

Remark. A special case of Theorem 2 is Theorem 2 in [3].

3

Let $\{X_t\}_{t \in T}$ be a family of topological spaces. Denote by $\Gamma$ the set of all finite subsets of $T$. For any $\alpha \in \Gamma$ put $X_\alpha = \prod_{t \in \alpha} X_t$. If $\alpha, \beta \in \Gamma$, $\alpha \supseteq \beta$ then $\pi_{\alpha \beta}$ denotes the projection $\pi_{\alpha \beta} : X_\alpha \to X_\beta$. Every $X_\alpha$ is a topological space with the product topology and every $\pi_{\alpha \beta}$ is a continuous mapping. Let $G$ be a weakly regular Abelian $l$-group.

Now we shall assume that we are given a consistent family of inner regular $G$-valued measures $\{\mu_t\}_{t \in T}$. Of course, regularity is taken with respect to the compact family of compact subsets of the corresponding space. Hence for every $\alpha \supseteq \beta$ and every $E \in \mathcal{R}_\beta$ ($\mathcal{R}_\beta$ is the domain of $\mu_\beta$) we have

$$\pi^{-1}_{\alpha \beta}(E) \in \mathcal{R}_\alpha, \quad \mu_\alpha(\pi^{-1}_{\alpha \beta}(E)) = \mu_\beta(E).$$
In this case the projective limit of the projective system \((X_\alpha, \mathcal{R}_\alpha, \mu_\alpha, \pi_\alpha)\) is 
\((X, \mathcal{R}, \mu, \pi_\alpha)\), where \(X = X_\alpha, \pi_\alpha\) is the projection \(\pi_\alpha : X \to X_\alpha, \mathcal{R} = \{\pi_\alpha^{-1}(E); E \in \mathcal{R}_\alpha, \alpha \in \Gamma\}\). 
\(\mu_\alpha(E) = \mu(\pi_\alpha^{-1}(E)) = \mu_\alpha(E)\). It is not difficult to prove that the definition of \(\mu\) is correct (\(\mu\) does not depend on the choice of \(\alpha\)), \(\mathcal{R}\) is a ring and that \(\mu\) is additive, monotone, \(\mu(\emptyset) = 0\). The only problem is whether \(\mu\) is \(\sigma\)-additive, i.e.

whether

\((X, \mathcal{R}, \mu, \pi_\alpha)\)

is the projective limit of the system in the category of measure spaces (see [1], [2]).

**Theorem 3.** Let \(G\) be a weakly regular, \(\sigma\)-complete, Abelian l-group. The function \(\mu\) defined above is a measure and \((X, \mathcal{R}, \mu, \pi_\alpha)\) is the projective limit in the category of measure spaces.

**Proof.** To prove that \(\mu\) is \(\sigma\)-additive it suffices to prove that \(\mu\) is upper semicontinuous in \(\emptyset\). Let \(\mathcal{C}_\alpha\) denote the family of all compact sets in \(X_\alpha\). Put

\[\mathcal{C} = \{\pi_\alpha^{-1}(E); E \in \mathcal{C}_\alpha, \alpha \in \Gamma\}.
\]

Evidently \(\mu\) is inner regular with respect to \(\mathcal{C}\). We prove that \(\mathcal{C}\) is a compact family.

Let \(C_n \in \mathcal{C}, C_n \supset C_{n+1}, C_n \neq \emptyset\) (\(n = 1, 2, \ldots\)). Then \(C_n = \pi_n^{-1}(D_n), D_n \in \mathcal{C}_{\alpha_n}\)

\((n = 1, 2, \ldots)\). The set \(\bigcup \alpha_n\) is countable. Put

\[\bigcup_{n=1}^{\infty} \alpha_n = \{t_1, t_2, t_3, \ldots\}.
\]

Consider the sequence

\[\{\pi(t_1)(C_n)\}_{n=1}^{\infty}.
\]

If \(t_1 \notin \alpha_n\), then \(\pi(t_1)(C_n) = X_{t_1}\). If \(t_1 \in \alpha_n\), then \(\{t_1\} \subset \alpha_n\), hence

\[\pi(t_1)(C_n) = \pi(t_1)\pi_n^{-1}(D_n) = \pi_{\alpha_n(t_1)}(D_n)
\]

and this is a compact subset of \(X_{t_1}\). Moreover, the sequence \(\{\pi(t_1)(C_n)\}_{n=1}^{\infty}\) is decreasing, therefore

\[\bigcap_{n=1}^{\infty} \pi(t_1)(C_n) \neq \emptyset.
\]

Denote by \(x_0^{t_1}\) an element of \(\bigcap_{n=1}^{\infty} \pi(t_1)(C_n)\) and repeat the procedure with the second coordinate \(t_2:\)

\[E_n = \pi(t_2)(C_n \cap \pi(t_1)^{-1}(\{x_0^{t_1}\}))\).
\]

Then \(E_n \supset E_{n+1}, E_n\) is closed \((n = 1, 2, \ldots)\) and \(E_n\) is compact if \(t_2 \in \alpha_n\). Hence

\[\bigcap_{n=1}^{\infty} E_n \neq \emptyset.
\]
Denote by \( x_1^0 \) an element of \( \bigcap_{n=1}^{\infty} E_n \). Repeating this procedure we obtain a sequence
\[
x_1^0, x_2^0, \ldots, x_k^0, \ldots
\]
such that
\[
x_k^0 \in \bigcap_{n=1}^{\infty} \pi_{(tn)}(C_n \cap \bigcap_{i=1}^{k-1} \pi_{(t(i))}^{-1}(\{x_i^0\})) \quad k = 1, 2, \ldots,
\]
hence to any \( n \) there is \( x \in C_n \) such that
\[
x_{t_1} = x_{t_1}^0, x_{t_2} = x_{t_2}^0, \ldots, x_{t_k} = x_{t_k}^0.
\]
Define \( x^0 \) by the following formula:
\[
(x^0)_t = x_t^0 \quad \text{if} \quad t \in \bigcup \alpha_n,
\]
\[
(x^0)_t = \text{an arbitrary element of } X_n \quad \text{if} \quad t \notin \bigcup \alpha_n.
\]
Now we assert that \( x^0 \in \bigcap_{n=1}^{\infty} C_n \).

Take arbitrary \( n \) and \( k \) such that \( \alpha_n \subset \{t_1, \ldots, t_k\} \). We know that there is \( x \in C_n \) such that
\[
x_{t_1} = x_{t_1}^0, x_{t_2} = x_{t_2}^0, \ldots, x_{t_k} = x_{t_k}^0.
\]
Put \( \alpha_n = \{t_{j_1}, \ldots, t_{j_m}\} \). Since \( x \in C_n \), \( \pi_{\alpha_n}(x) \in D_n \) hence
\[
(x_{t_{j_1}}^0, \ldots, x_{t_{j_m}}^0) = (x_{t_{j_1}}, \ldots, x_{t_{j_m}}) \in D_n.
\]
But it follows that \( \pi_{\alpha_n}(x^0) \in D_n \), i.e. \( x^0 \in \pi_{\alpha_n}^{-1}(D_n) = C_n \).

We have proved that \( \mathcal{C} \) is a compact system. By Theorem 1 \( \mu \) is upper continuous in \( \emptyset \), i.e. \( \mu \) is a measure.

References


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