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Časopis pro pěstování matematiky, Vol. 101 (1976), No. 4, 350--359

Persistent URL: http://dml.cz/dmlcz/117931

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SETS OF $\sigma$-POROSITY AND SETS OF $\sigma$-POROSITY ($q$)

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(Received July 11, 1975)

1. INTRODUCTION

The notion of a set of $\sigma$-porosity was defined by E. P. DOLŽENKO [1]. There exists a number of theorems in the theory of cluster sets which use this notion. (See [1], [2], [3], [4], [5].) It is easy to see that any set of $\sigma$-porosity is of the first category and of measure zero. The existence of a set of the first category and of measure zero which is not of $\sigma$-porosity is claimed without a proof in [1]. In the present article we shall prove this result.

N. YANAGIHARA [2] defined and used the notion of a set of $\sigma$-porosity ($q$), $0 < q \leq 1$, which coincides with the notion of a set of $\sigma$-porosity for $q = 1$. We shall prove that the notions of a set of $\sigma$-porosity ($q$) and $\sigma$-porosity ($p$) coincide for any $p, q, 0 < p, q < 1$.

The main aim of the present article is to prove the results mentioned above and some other results on the sets of $\sigma$-porosity ($q$) (in our notation on sets of ($x^q$)-$\sigma$-porosity) in Euclidean spaces.

We shall generalize the notion of a set of $\sigma$-porosity ($q$) and we shall formulate some results in a general metric space in order to clarify the proofs.

2. DEFINITIONS

Let $(P, d)$ be a metric space. Then we define:

2.1. The open sphere with the centre $x \in P$ and the radius $r > 0$ is denoted by $K(x, r)$.

2.2. Let $M \subseteq P$, $x \in P$, $R > 0$. Then we denote the supremum of the set \{$r > 0$; for some $z \in P$, $K(z, r) \subseteq K(x, R)$ and $K(z, r) \cap M = \emptyset$\} by $\gamma(x, R, M)$.

2.3. Let $K(x, r) \subseteq P$. Let $f$ be an arbitrary function. Then we put $f \ast K(x, r) = K(x, f(r))$ if $f(r) > 0$. 

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2.4. Let $M \subset P$. Let $f$ be an arbitrary function. Then we put 
\[ S(f, r, M) = \bigcup \{ f \ast K; K \cap M = \emptyset, K = K(x, \sigma), \sigma < r \text{ and } f(\sigma) > 0 \}. \]

2.5. We denote by $G$ (resp. $G_1$, resp. $G_2$) the system of all real functions which are increasing and continuous (resp. for which $\infty > g'(x) \geq 1$, resp. for which $\infty > g'(x) \geq 1$ and $g(x) > x$) on $(0, \delta)$ for some $\delta > 0$.

2.6. We denote by $G_3$ the system of all functions $g \in G$ such that for any $A > 0$ and $\varepsilon > 1$ there exists an integer $r$ and $\delta > 0$ such that 
\[ (eg) \circ \ldots \circ (eg) \circ (x) \geq A g(x) \text{ for } 0 < x < \delta. \]

2.7. Let $f \in G$, $M \subset P$, $x \in P$. Then we say that $x$ is a point of $(f)$-porosity of $M$ if 
\[ \limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon} f(y(x, R, M)) > 0. \]

2.8. Let $f \in G$, $M \subset P$, $x \in P$, $c > 0$. Then we say that $x$ is a point of $(f, c)$-porosity of $M$ if 
\[ \limsup_{\varepsilon \to 0^+} \frac{1}{\varepsilon} f(y(x, R, M)) \geq c. \]

2.9. Let $g \in G$, $H \subset G$, $M \subset P$, $x \in P$. Then we say that $x$ is a point of $(g)$-porosity of $M$ if $x \in \bigcap \{ S(g, r, M); r > 0 \}$. We say that $x$ is a point of $(H)$-porosity of $M$ if it is a point of $(h)$-porosity of $M$ for any $h \in H$.

2.10. Let $V$ be one of the symbols $(f)$, $(f, c)$, $(h)$, $(H)$. Let $M \subset P$, $N \subset P$.

Then we say that $M$ is of $V$-porosity if any point $x \in M$ is a point of $V$-porosity of $M$. We say that $N$ is a set of $V$-$\sigma$-porosity if it is the union of a sequence of sets of $V$-porosity.

2.11. We shall write "porosity" instead of "$(x)$-porosity" and "$\sigma$-porosity" instead of "$(x)$-$\sigma$-porosity".

Let us note:

2.12. The notions of a set of $(x^\circ)$-porosity and of a set of $(x^\circ)$-$\sigma$-porosity coincide with the notions of $N$. YANAGIHARA of a set of porosity $(q)$ and of a set of $\sigma$-porosity $(q)$.

2.13. Let $V$ be one of the symbols $(f)$, $(f, c)$, $(h)$, $(H)$. Then the system of all sets of $V$-porosity is an ideal of sets and the system of all sets of $V$-$\sigma$-porosity is a $\sigma$-ideal of sets.

2.14. A point $x \in \mathbb{R}^k$ is a point of $(x, 1/2)$-porosity of a set $M \subset \mathbb{R}^k$ iff there exists a sequence of spheres \( \{ K(s_n, r_n) \} \) such that \( \lim s_n = x \), \( \lim g(x, s_n)/r_n = 1 \) and \( K(s_n, r_n) \cap M = \emptyset \) for $n = 1, 2, \ldots$. 

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2.15. Evidently, we may always write \((af, ac)\) instead of \((f, c)\) if \(a > 0\).

2.16. A point \(x \in P\) is a point of \(<h>-porosity\) of a set \(M \subset P\) iff there exists a sequence of spheres \(\{K(s_n, r_n)\}\) such that \(\lim_{n \to \infty} s_n = x\), \(K(s_n, r_n) \cap M = \emptyset\) and \(h(r_n) > q(x, s_n)\) for \(n = 1, 2, \ldots\).

2.17. Let \(V\) be one of the symbols \((f), (f, c), \langle h\rangle, \langle H\rangle\). Let \(x \in P, M \subset P\). Then the point \(x\) is a point of \(V\)-porosity of the set \(M\) iff it is a point of \(V\)-porosity of the set \(M\).

3. SEVERAL LEMMAS

3.1. Lemma. Let \((P, g)\) be a metric space, \(M \subset P, f \in G\). Then:
(i) If \(x \in P\) is a point of \((f, 2)-porosity\) of \(M\) then it is a point of \(\langle f\rangle\)-porosity of \(M\).
(ii) If \(x \in M\) is a point of \(\langle f\rangle\)-porosity of \(M\) then it is a point of \((2f, 1)-porosity\) of \(M\).

Proof. The assertion (i) immediately follows from the continuity of \(f\) on \((0, \delta)\) and from the definitions. The assertion (ii) follows from the fact that if \(K(y, h) \subset \subset P - M, x \in f \ast K(y, h)\) and \(h\) is sufficiently small then \(f(y(x, R, M)) > R/2\) where \(R = 2q(x, y)\).

3.2. Lemma. Let \(g \in G_1\). Then if \(d > 0\) is a sufficiently small number, the relations \(z \in g \ast I, I = (t - r, t + r)\) and \(I \subset J = (u - d, u + d)\) imply \(z \in g \ast J\).

Proof. Let \(d < \delta\), where \(\delta\) is the number from the definition of the system \(G_1\) (see 2.5). Then

\[q(z, u) \leq q(z, t) + d - r < g(r) + d - r \leq g(d)\]

and therefore \(z \in g \ast J\).

3.3. Lemma. Let \(M \subset (a, b)\) be a nowhere dense set. Let \(g \in G_2\). Let \(\{I_n\}_{n=1}^{\infty}\) be a sequence of pairwise disjoint open intervals such that \((a, b) - \overline{M} = \bigcup_{n=1}^{\infty} I_n\). Let \(H\) be the set of all endpoints of intervals \(I_n\). Let \(P\) be the set of all points of \(\langle g\rangle\)-porosity of \(M\) which lie in \(\overline{M} \cap (a, b)\). Then

\[P = (H \cup \limsup_{n \to \infty} g \ast I_n) \cap (a, b)\]

Proof. Let \(z \in (H \cup \limsup_{n \to \infty} g \ast I_n) \cap (a, b)\). If \(z \in H\), then \(z \in P\), since \(g(x) > x\) for sufficiently small \(x\). If \(z \in \limsup_{n \to \infty} g \ast I_n\), then evidently \(z \in P\). Let \(y \in P - H\). Then 2.16 and 3.2 clearly imply \(z \in \limsup_{n \to \infty} g \ast I_n\).
3.4. Lemma. Let $H \subset G, f \in G, c > 0$. Let $n$ be an integer and $M \subset R^1$. Put $N = M \times R^n$. Then $N$ is a set of $(f, c)\sigma$-porosity or of $(f, c)$-porosity, or of $(f)\sigma$-porosity, or of $(f)$-porosity, or of $(H)\sigma$-porosity, or of $(H)$-porosity in the space $R^{n+1}$ iff $M$ is of the same type as a subset of $R^1$.

Proof. We shall prove only the part concerning $(f, c)\sigma$-porosity, the proofs of the other parts being quite similar. The implication "if" follows from the fact that

$$
\gamma((x, y), r, A \times R^n) \geq \gamma(x, r, A) \quad \text{for any } x \in R^1, y \in R^n, A \subset R^1 \text{ and } r > 0.
$$

Now we shall prove the implication "only if". Let $N = \bigcup_{k=1}^{\infty} N_k$ where any $N_k$ is a set of $(f, c)$-porosity. Let $\{B_t\}_{t=1}^{\infty}$ be a basis of open sets in $R^n$. Denote by $A_{k,t}$ the set of all points $x \in M$ for which the set $\{z; (x, z) \in N_k\}$ is dense in $B_t$. Clearly $M = \bigcup_{k,t} A_{k,t}$ and therefore it is sufficient to prove that each set $A_{k,t}$ is of $(f, c)$-porosity. Let $x \in A_{k,t}$ and $z \in B_t$ be such that $(x, z) \in N_k$. Clearly for any $r > 0$ such that $K(z, r) \subset C B_n$, the inequality $\gamma((x, z), r, N_k) \leq \gamma(x, r, A_{k,t})$ holds. Since $N_k$ is a set of $(f, c)$-porosity in $R^{n+1}$, the set $A_{k,t}$ is a set of $(f, c)$-porosity in $R^1$.

3.5. Lemma. Let $P$ be a metric space and $f \in G$. Let $A \subset P$ be a set of $(f)\sigma$-porosity. Then $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n$ is a set of $(f, c_n)$-porosity for some $c_n > 0$, $n = 1, 2, \ldots$.

Proof. Let $A = \bigcup_{i=1}^{\infty} B_i$ where each set $B_i$ is a set of $(f)$-porosity. For any $i$ let $B_{i,k}$ be the set of all points $x \in B_i$ which are points of $(f, 1/k)$-porosity of the set $B_i$. Clearly $B_i = \bigcup_{k=1}^{\infty} B_{i,k}$ and each set $B_{i,k}$ is a set of $(f, 1/k)$-porosity. Now it is sufficient to order the sets $B_{i,k}$ in a sequence $\{A_n\}_{n=1}^{\infty}$.

4. SOME AFFIRMATIVE RESULTS

In the present part we shall prove that some properties like $\sigma$-porosity are equivalent with other, seemingly weaker properties of this type. We use only one method which is contained in the following basic proposition.

4.1. Proposition. Let $h \in G, f \in G$. Let there exist an integer $n$ and $\delta > 0$ such that

$$
(1) \quad h^{(n)}(x) = h \circ \ldots \circ h(x) \geq f(x) \quad \text{for } 0 < x < \delta.
$$

Let $P$ be a metric space and let $M \subset P$ be a set of $(f)\sigma$-porosity. Then $M$ is a set of $(h)\sigma$-porosity.
Proof. It is clearly sufficient to prove that if $A$ is a set of $\langle f \rangle$-porosity then it is a set of $\langle h \rangle$-$\sigma$-porosity. Put $C_k = A \cap \bigcap_{r > 0} S(h^{(k)}, r, A)$, (see 2.4). Then $A \subseteq \bigcup_{k=0}^{n} (C_k - C_{k-1}) \cup C_1$. Since obviously $C_1$ is a set of $\langle h \rangle$-porosity it is sufficient to prove that $C_k - C_{k-1}$ is a set of $\langle h \rangle$-$\sigma$-porosity for $k = 2, ..., n$. Put $T_{k,m} = C_k - S(h^{(k-1)}, 1/m, A)$ for $k = 2, ..., n$ and $m = 1, 2, ...$. Since clearly $C_k - C_{k-1} = \bigcup_{m=1}^{\infty} T_{k,m}$, it is sufficient to prove that any set $T_{k,m}$ is a set of $\langle h \rangle$-porosity.

Let $z \in T_{k,m}, r > 0$. Then there exists an open sphere $K(y, t)$ such that $t < \min (1/m, r)$, $K(y, t) \cap A = \emptyset$ and $z \in h^{(k)} * K(y, t)$. Put $K = h^{(k-1)} * K(y, t)$. Then $z \in h * K$ and $K \cap T_{k,m} = \emptyset$ since $K \subseteq S(h^{(k-1)}, 1/m, A)$. Since the radius of the sphere $K$ is arbitrarily small provided $r$ is sufficiently small, $z$ is a point of $\langle h \rangle$-porosity of the set $T_{k,m}$. Therefore $T_{k,m}$ is a set of $\langle h \rangle$-porosity. Thus the proof is complete.

4.2. Proposition. Let $h \in G, f \in G$. For any $B > 0$, let there exist $A > 0, \delta > 0$ and an integer $r$ such that

$$
(Ah) \circ \cdots \circ (Ah)(x) \geq B f(x) \quad \text{for } 0 < x < \delta.
$$

Let $P$ be a metric space and let $M \subseteq P$ be a set of $(f)$-$\sigma$-porosity. Then $M$ is a set of $(h)$-$\sigma$-porosity.

Proof. By 3.5, $M = \bigcup_{m=1}^{\infty} M_m$ where $M_m$ is a set of $(f, c_m)$-porosity, $c_m > 0$. By 2.15 and 3.1 $M_m$ is a set of $\langle 2f/c_m \rangle$-porosity. By 4.1 and (2) it is a set of $\langle Ah \rangle$-$\sigma$-porosity for some $A > 0$. Therefore by 2.15 and 4.1 it is a set of $(h)$-$\sigma$-porosity. Consequently $M$ is of $(h)$-$\sigma$-porosity.

4.3. Theorem. Let $0 < q < p < 1$ and let $M$ be a subset of a metric space. Then $M$ is a set of $(x^q)$-$\sigma$-porosity iff it is a set of $(x^p)$-$\sigma$-porosity.

Proof. Let $B > 0$. Then the inequality (2) from Proposition 4.2 holds for $A = 1, h = x^p, f = x^q$, an integer $r$ such that $p^r < q$ and for a sufficiently small $\delta > 0$. Therefore the statement of the theorem follows from 4.2.

4.4. Proposition. Let $P$ be a metric space and $g \in G_3$ (see 2.6). Let $M \subseteq P$ be a set of $(g)$-$\sigma$-porosity and $0 < c < \frac{1}{2}$. Then $M$ is a set of $(g, c)$-$\sigma$-porosity.

Proof. By 3.5 it is sufficient to prove that any set $N$ of $(g, a)$-porosity is a set of $(g, c)$-$\sigma$-porosity. By 3.1, $N$ is a set of $\langle 2g/a \rangle$-porosity. Put $A = 2/a$ and $e = \frac{1}{2c}$. Let $r$ be the integer from 2.6. Then the inequality (1) from 4.1 holds for $f = 2g/a, h = g/2c$ and for sufficiently small $\delta > 0$. Therefore by 4.1, $N$ is a set of $\langle g/2c \rangle$-$\sigma$-porosity and consequently it is a set of $(g, c)$-$\sigma$-porosity.

Since obviously $x^q \in G_3$ for $0 < q \leq 1$, we have

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4.5. **Theorem.** Let $P$ be a metric space, $0 < q \leq 1$, $0 < c < \frac{1}{2}$. Then a subset of $P$ is a set of $(x^q)$-a-porosity iff it is a set of $(x^q, c)$-a-porosity.

5. SOME NEGATIVE RESULTS

In the present part we shall prove that some properties like $\sigma$-porosity are not equivalent with the others. We use only one method which is contained in the following basic proposition.

5.1. **Proposition.** Let $f \in G$ and $H \subset G_2$ (see 2.5). Let there exist a sequence $\{h_i\}_{n=1}^{\infty}$ of functions from $H$ and a sequence of positive numbers $\{e_n\}_{n=1}^{\infty}$ such that

\[ h_n \circ \ldots \circ h_1(x) < f(x) \quad \text{for} \quad 0 < x < e_n. \]

Then in any Euclidean space there exists a perfect set $F$ of $\langle f \rangle$-porosity which is not a set of $\langle H \rangle$-$\sigma$-porosity.

Along with 5.1, we shall prove the following proposition.

5.2. **Proposition.** Let $g \in G$ and $\lim_{x \to 0^+} x|g(x)| = 0$. Then in any Euclidean space there exists a perfect set $F$ of $(g, 1)$-porosity and of measure zero which is not of $\sigma$-porosity.

**Proof.** 3.4 implies that it is sufficient to construct a set $F$ on the line. Let $\{k_i\}_{i=1}^{\infty}$ be an increasing sequence of integers such that $k_1 = 1$. Our construction depends on this sequence. For a proof of 5.1, the sequence $\{k_i\}$ may be chosen in an arbitrary way but for a proof of 5.2 we must choose it in a special way. Given $\{k_i\}$ define a sequence $\{s_p\}_{p=1}^{\infty}$ by the relations $k_{s_p} \leq p \leq k_{s_p+1}$. We may and will assume that $\lim_{n \to \infty} e_n = 0$.

From the segment $\langle 0, 1 \rangle$ we shall delete in the $k$-th step a finite number of pairwise disjoint intervals, $D$-intervals of the order $k$. The points from $\langle 0, 1 \rangle$ not contained in any $D$-interval will form the set $F$. For any integer $k$ we shall define a system of remaining intervals ($R$-intervals) of the order $k$. Any $R$-interval will be closed. The system of all $R$-intervals of the order $k$ and of all $D$-intervals of orders $j \leq k$ will form a covering of $\langle 0, 1 \rangle$ and any two members of this system will have disjoint interiors.

Define the $D$-intervals and the $R$-intervals by induction:

1. A $D$-interval of the order 1 does not exist. As the system of all $R$-intervals of the order 1, let us choose any covering of $\langle 0, 1 \rangle$ by closed intervals of a length smaller than $e_2$ such that any two its members have disjoint interiors.

2. Let $k$ be an integer. Let $D$-intervals and $R$-intervals of all orders smaller than $k + 1$ be defined. Let $R_1, \ldots, R_{i_k}$ be all $R$-intervals of the order $k$. For $j = 1, \ldots, i_k$
define an open interval \( D_j \subset R_j \) by the relation \((h_{sk+1} \circ \ldots \circ h_1) \ast D_j = R_j\). Define the system of all \( D \)-intervals of the order \( k + 1 \) as the system \( D_1, \ldots, D_{bk} \). The endpoints of the intervals \( D_j \) and \((h_1 \circ \ldots \circ h_1) \ast D_j, j = 1, \ldots, i_k, t = 1, \ldots, s_k + 1 \) divide \((0, 1)\) to a finite number of closed intervals. Let \( A_1, \ldots, A_{bk} \) be all of these intervals which are disjoint with each \( D \)-interval of the order \( k + 1 \).

For \( 1 \leq r \leq b_k \) let \( C_r \) be a system of closed intervals of a length smaller than \( e_{sk+1} + 1 \) such that \( \bigcup \{X; X \in C_r\} = A_r \) and any two members of \( C_r \) have disjoint interiors. Define the system of all \( R \)-intervals of the order \( k + 1 \) as the system \( \bigcup_{r=1}^{b_k} C_r \).

The following assertions are easily verified:

(i) \( F \) is a perfect set of \( \langle f \rangle \)-porosity.

(ii) Let \( R \) be an \( R \)-interval of an order \( k \) and let \( m \leq s_k \) be an integer. Then the set \( R = \bigcup \{(h_m \circ \ldots \circ h_1) \ast D; \ D \subset R \) is a \( D \)-interval\} is a nonempty perfect set. If a contiguous interval of this set lies in \( R \) then it is of the form \((h_m \circ \ldots \circ h_1) \ast D\), where \( D \subset R \) is a \( D \)-interval.

(iii) Let \( D \) be a \( D \)-interval of an order \( k \) and let \( m \leq s_{k-1} + 1 \) be an integer. Let \( R \) be a \( R \)-interval such that \( \text{Int} R \cap D = \emptyset \). Then either \( \text{Int} R \subset (h_m \circ \ldots \circ h_1) \ast D \) or \( R \cap (h_m \circ \ldots \circ h_1) \ast D = \emptyset \).

Now suppose that \( F \) is a set of \( \langle H \rangle \)-\( \sigma \)-porosity. Then \( F = \bigcup_{i=1}^{\infty} P_i \) where each \( P_i \) is a set of \( \langle H \rangle \)-porosity. We shall define a sequence \( \{F_i\}_{i=0}^{\infty} \) of nonempty perfect sets such that \( F \supset F_{i-1} \supset F_i \) and \( F_i \cap P_i = \emptyset \) for \( i = 1, 2, \ldots \). The existence of such a sequence yields a contradiction since it implies that there exists a point \( x \in \cap_{i=0}^{\infty} F_i \subset F \) which does not lie in \( \bigcup_{i=1}^{\infty} P_i = F \). Each set \( F_i \) will have the form

\[
F_i = R_i - \bigcup_{i=1}^{\infty} \left\{(h_i \circ \ldots \circ h_1) \ast D; \ D \subset R_i \right\} \text{ is a } D \text{-interval},
\]

where \( R_i \) is an \( R \)-interval of an order \( j \geq k_{i+1} \) and \((h_0 \circ \ldots \circ h_1) \ast D = D\). By (ii) any set of the form (4) is a nonempty perfect set.

Define the sets \( F_i \) by induction:

A. Put \( F_0 = R_0 \cap F \) where \( R_0 \) is an \( R \)-interval of the order 1.

B. Suppose that we have defined the set \( F_i \). We shall distinguish two cases:

B 1. \( F_i \notin P_{i+1} \). Then define \( R_{i+1} \) as an \( R \)-interval of an order \( j \geq k_{i+2} \) such that \( R_{i+1} \cap P_{i+1} = \emptyset \) and \( R_{i+1} \cap F_i \) is an infinite set. Define the set \( F_{i+1} \) by (4).

B 2. \( F_i \in P_{i+1} \). Then any point of \( P_{i+1} \) is a point of \( \langle h_{i+1} \rangle \)-porosity of \( F_i \). Therefore by 3.3 and (ii) any point \( x \in \text{Int} R_i \cap P_{i+1} \) lies in an interval of the form

\[
(h_{i+1} \circ \ldots \circ h_1) \ast ((h_0 \circ \ldots \circ h_1) \ast D) = (h_{i+1} \circ \ldots \circ h_1) \ast D,
\]

where \( D \subset R_i \) is a \( D \)-interval. Therefore the nonempty perfect set \( A = R_i - \bigcup \{(h_{i+1} \circ \ldots \circ h_1) \ast D; \ D \subset R_i\} \) and the set \( \text{Int} R_i \cap P_{i+1} \) are disjoint. Define \( R_{i+1} \).
as an $R$-interval of an order $j \geq k_i + 2$ such that $R_{i+1} \subset \text{Int} R_i$ and $R_{i+1} \cap A$ is an infinite set. Then define the set $F_{i+1}$ by (4). Since (iii) implies $F_{i+1} = R_{i+1} \cap A$ we have $F_{i+1} \cap P_{i+1} = 0$. Thus the proof of 5.1 is complete.

To prove 5.2 put $f = g/2$ and $H = \{6x\}$. Then the assumptions of 5.1 are obviously fulfilled. If we denote by $m_i$ the measure of the union of all $R$-intervals of the order $k_i$, then evidently

$$m_{i+1} = m_i (1 - 1/6^{i+1})^{k_{i+1} - k_i}$$

Therefore there exists a sequence $\{k_i\}_{i=1}^\infty$ such that $\lim_{i \to \infty} m_i = 0$ and consequently $\mu F = 0$. The set $F$ is a set of $(g/2)$-porosity and therefore it is of $(g, 1)$-porosity. On the other hand, $F$ is not a set of $(6x)$-$\sigma$-porosity and therefore it is not a set of $(3x, 1)$-$\sigma$-porosity. Now 4.5 implies that $F$ is not a set of $\sigma$-porosity.

5.3. Proposition. Let $h \in G_3, f \in G_1$. Let there exist $B > 0$ such that for any $A > 0$ and any integer $r$ there exists $\delta > 0$ such that

$$\left( A h \right)^{r} \left( A h \right)(x) < B f(x) \quad \text{for} \quad 0 < x < \delta.$$ 

Then in any Euclidean space there exists a perfect set of $(f)$-porosity which is not of $(h)$-$\sigma$-porosity.

Proof. By 5.1, in any Euclidean space there exists a perfect set $F$ of $(bf)$-porosity which is not of $(6h)$-$\sigma$-porosity. Thus $F$ is a set of $(f)$-porosity but not of $(3h, 1)$-$\sigma$-porosity and by 4.4 it is not of $(h)$-$\sigma$-porosity.

The following theorem is a consequence of 5.2.

5.4. Theorem. Let $0 < q < 1$. Then in any Euclidean space there exists a perfect set $F$ of $(x^q, 1)$-porosity and of measure zero which is not of $\sigma$-porosity.

The existence of a perfect set of $(x^q)$-porosity which is not of $\sigma$-porosity follows also from the following easy theorem.

5.5. Theorem. Let $0 < q < 1$. Then in any Euclidean space there exists a perfect set $D$ of $(x^q)$-porosity and of positive Lebesgue measure.

Proof. 3.4 implies that it is sufficient to construct the set $D$ on the line. We shall define a sequence of sets such that $S_k$ contains $2^k$ disjoint closed intervals:

1. $S_0 = \{0, 1/2\}$.

2. Suppose that we have defined $S_k = \{I_1, \ldots, I_{2^k}\}$. For $j = 1, \ldots, 2^k$ define closed disjoint intervals $I'_j, I''_j$ such that

$$x^q * (I_j - (I'_j \cup I''_j)) = \text{Int} I_j.$$

Put $S_{k+1} = \{I'_1, I''_1, \ldots, I'_2^2, I''_2^2\}$. Put $D_k = \bigcup \{I; I \in S_k\}$ and $D = \bigcap_{k=0}^\infty D_k$. The set $D$ is
clearly a perfect set of \((x^q)\)-porosity. We have
\[
\mu(I_j' \cup I_j'') = \mu I_j (1 - 2^{1 - 1/q} (\mu I_k)^{1/q - 1}).
\]
Since \(\mu I_j < 1/2^{n+1}\), we have
\[
\mu(I_j' \cup I_j'') > \mu I_j (1 - 2^{(1 - 1/q)(n+2)}).
\]
If we denote \(\mu D_k = m_k\), we have
\[
m_{n+1} > m_n (1 - 2^{(1 - 1/q)(n+2)})
\]
and therefore
\[
m_{n+1} > \frac{1}{2} \prod_{k=0}^{n} (1 - 2^{(1 - 1/q)(k+2)})
\]
and
\[
\mu D \geq \frac{1}{2} \prod_{k=0}^{\infty} (1 - 2^{(1 - 1/q)(k+2)}) > 0.
\]
Thus the proof is complete.

The following theorem justifies the complicated form of 5.1.

5.6. Theorem. In any Euclidean space there exists a perfect set \(F\) of porosity which is not a set of \((x, 1/2)\)-\(\sigma\)-porosity.

Proof. Let \(H = \{ax; a > 1\}\). For an integer \(n\) put \(h_n = (1 + 1/n^2) x\). Put \(c = \prod_{k=1}^{\infty} (1 + 1/k^2)\) and \(f(x) = 2cx\). Then the assumption (3) from 5.1 is obviously fulfilled and therefore in any Euclidean space there exists a perfect set \(F\) of \(\langle 2cx \rangle\)-porosity which is not a set of \(\langle H \rangle\)-\(\sigma\)-porosity. The set \(F\) is clearly a set of porosity but not of \((x, 1/2)\)-\(\sigma\)-porosity since a set is of \((x, 1/2)\)-porosity iff it is of \(\langle H \rangle\)-porosity.

5.7. Theorem. Let \(0 < q < 1\). Then in any Euclidean space there exists a perfect set \(D\) which is not a set of \((x^q)\)-\(\sigma\)-porosity.

Proof. The theorem immediately follows from 5.3 if we put \(h = x^q\), \(f = (\log (1/x))^{-1}\), \(B = 1\).

6. SOME OPEN PROBLEMS

6.1. Problem. Does there exist a (perfect) set on the line of the first category and of measure zero which is not a set of \((x^q)\)-\(\sigma\)-porosity for \(0 < q < 1\)?

6.2. Problem. Does there exist \(f \in G\) such that any (perfect) set on the line of measure zero and of the first category is a set of \((f)\)-\(\sigma\)-porosity?
References


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