William A. Webb

On the Diophantine equation \( \frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} \)

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ON THE DIOPHANTINE EQUATION

\[ \frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} \]

WILLIAM A. WEBB, Pullman

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Introduction. Given positive integers \( k \) and \( n \), and nonzero integers \( a_1, a_2, \ldots, a_r \), consider the equation

\[ \frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \ldots + \frac{a_r}{x_r} \]

where the \( x_i \) are positive integers such that \((a_i, x_i) = 1\). Let \((1')\) denote the same equation where the \( x_i \) can be any nonzero integer. In the special case \( a_1 = a_2 = \ldots = a_r = 1 \), the so-called Egyptian or unit fractions, these equations have been extensively studied.

Let \( \lambda = \lambda(k; a_1, a_2, \ldots, a_r) \) be the largest integer \( n \) for which the equation \((1)\) has no solution. If \((1)\) is unsolvable for infinitely many values of \( n \), set \( \lambda = \infty \). If \((1)\) is solvable for all positive \( n \), set \( \lambda = 0 \). Also, define \( \lambda' \) similarly with respect to equation \((1')\). Very little is known about precise values of \( \lambda \) and \( \lambda' \), even in special cases.

In this paper we will consider solutions of equation \((1)\) with particular attention to the cases \( r = 2 \) and \( 3 \). The principal result obtained is a lower bound for \( \lambda \) and \( \lambda' \) when \( r = 3 \), and \( k \) is large.

Preliminary Results and the Case \( r = 2 \). If \( p \) is a prime, and \( p \mid (a_1, \ldots, a_r) \), then \((1)\) is not solvable for \( n = p^s \) and all \( s \) sufficiently large, since \((x_1, p) = 1 \). Hence, if \((a_1, \ldots, a_r) \neq 1, \lambda = \infty \) and so we assume henceforth that \((a_1, \ldots, a_r) = 1 \).

The case \( r = 1 \) is trivial and will not be mentioned again.

The following result gives necessary and sufficient conditions for \((1)\) to be solvable when \( r = 2 \).

Theorem 1. The equation

\[ \frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2} \]

is solvable in positive integers \( x_1, x_2 \) such that \((a_1, x_1) = 1 = (a_2, x_2)\), if and
only if there exist positive divisors \( d_1 \) and \( d_2 \) of \( n \) such that \( a_1d_1 + a_2d_2 = kt \) for some positive integer \( t \) such that \( (a_1, a_2, t) = 1 \), and \( (n/d_1, a_1) = (n/d_2, a_2) = 1 \).

Proof. If the conditions of the theorem are satisfied, then let \( x_i = tn/d_i \). We then have
\[
\frac{a_1}{x_1} + \frac{a_2}{x_2} = \frac{a_1d_1}{tn} + \frac{a_2d_2}{tn} = \frac{kt}{tn} = \frac{k}{n}
\]
and \( (a_t, x_i) = (a_t, tn/d_i) = 1 \) by the hypotheses.

Now suppose that (2) is satisfied by \( x_1 \) and \( x_2 \) such that \( (a_1, x_1) = (a_2, x_2) = 1 \).

Also, assume \( (k, n) = 1 \). Let \( d = (x_1, x_2) \), \( x_i = dX_i \) and \( t = (d, a_1X_2 + a_2X_1) \).

Then
\[
(3) \quad \frac{k}{n} = \frac{a_1x_2 + a_2x_1}{x_1x_2} = \frac{a_1X_2 + a_2X_1}{dX_1X_2} = \frac{(a_1X_2 + a_2X_1)/t}{(d/t)X_1X_2}
\]
Since \( (X_1, X_2) = 1 \) and \( (a_t, X_1) = 1, (a_1X_2 + a_2X_1, X_1X_2) = 1 \). This, together with \( ((a_1X_2 + a_2X_1)/t, d/t) = 1 \), implies that the right hand fraction in (3) is reduced, and so \( k = (a_1X_2 + a_2X_1)/t \) and \( n = (d/t)X_1X_2 \).

Therefore, letting \( d_1 = X_2 \) and \( d_2 = X_1 \), we have immediately that \( d_i \mid n \) and \( a_1d_1 + a_2d_2 = kt \). Also, \( (n/d_1, a_1) = (dX_1/t, a_1) = (x_1/t, a_1) = 1 \). Similarly \( (n/d_2, a_2) = 1 \). Finally, \( (a_t, x_i) = 1 \) which implies \( (a_t, d) = 1 \) and hence \( (a_t, t) = 1 \), which gives us \( (a_1a_2, t) = 1 \).

If \( (k, n) = b > 1 \), apply the above argument to \( K = k/b \) and \( N = n/b \) and then use divisors \( D_i = bd_i \).

We are now ready to consider \( \lambda(k; a_1, a_2) \) in more detail. We have already noted that we must have \( (a_1, a_2) = 1 \). It is also obvious that \( (k, a_1a_2) \neq 1 \) implies \( \lambda = \infty \), and so we will also assume \( (k, a_1a_2) = 1 \) for the rest of this section. Finally, \( \lambda = \infty \) if both \( a_1 \) and \( a_2 \) are negative, so without loss of generality \( a_1 > 0 \).

Theorem 2. Let \( (a_1, a_2) = (k, a_1a_2) = 1 \) and \( a_1 > 0 \). Then \( \lambda(k; a_1, a_2) = \infty \) unless

(i) \( k = 1 \) or \( 2 \) and \( a_2 \geq -1 \)

or

(ii) \( k > 2, a_2 = -1 \) and \( a_1 \neq 1 \) has the property that all primes dividing \( a_1 \) are \( \equiv 1 \) (mod \( k \)).

In these cases \( \lambda = 0 \), except that \( \lambda(1; 1, -1) = 1 \) and \( \lambda(2; 1, -1) = 2 \).

Proof. Write \( n = A_1A_2m \) where \( A_i \) is the largest divisor of \( n \) containing only primes which divide \( a_i \). The property mentioned in (ii) above will be called property \( P \).

If \( a_2 < -1 \), then there is a prime \( p \) which divides \( a_2 \). Then by Theorem 1, equation (2) is not solvable if \( n = p^s \) for \( s \) sufficiently large. The conditions \( (n/d_2, a_2) = 1 \) and \( ((a_1d_1 + a_2d_2)/k, a_1a_2) = 1 \) imply \( d_2 = p^s \) and \( d_1 = 1 \), respectively. Thus \( t = (a_1 + a_2p^s)/k < 0 \) for \( s \) sufficiently large. Therefore \( \lambda = \infty \) if \( a_2 < -1 \).
If $k > 2$ and either $a_1$ or $a_2$ does not have property $P$, let $p \nmid 1 \pmod{k}$ be a divisor of $a_1$. (We may suppose $a_1$ is divisible by $p$ and not $a_2$ since $a_2 = -1$ is only remaining case where $a_2 < 0$.) There are now infinitely many values of $s$ for which (2) is not solvable with $n = p^s$. Applying Theorem 1, just as above we find $d_1 = p^s$ and $d_2 = 1$, and $k$ does not divide $a_1p^s + a_2$ for infinitely many values of $s$. Thus $\lambda = \infty$ in this case also.

If $k > 2$, $a_2 > 0$ and both $a_1$ and $a_2$ have property $P$, then again $\lambda = \infty$ since (2) is not solvable for all primes $q \equiv 1 \pmod{k}$. In applying Theorem 1 we find $a_1d_1 + a_2d_2 \equiv 1 + 1 \equiv 0 \pmod{k}$.

If $k > 2$, $a_1 = 1$ and $a_2 = -1$ we apply Theorem 1 to $n = p$, a prime $\equiv 1 \pmod{k}$. Clearly none of the cases for $a_1d_1 + a_2d_2 = d_1 - d_2$ yield a positive integer divisible by $k$. Hence $\lambda = \infty$.

The only remaining case for $k > 2$ is $a_2 = -1$ and $a_1 \pm 1$ having property $P$. Write $n = A_1m$ and apply Theorem 1 with $d_1 = A_1$ and $d_2 = 1$. Then $a_1d_1 + a_2d_2 = A_1 - 1 \equiv 1 - 1 \equiv 0 \pmod{k}$ and $A_1A_1 - 1 > 0$. The other conditions of the theorem are satisfied since $(A_1, A_1A_1 - 1) = 1$, $(m, A_1) = 1$ and $(n, -1) = 1$. Therefore equation (2) is solvable for all $n$, and so $\lambda = 0$. The case $k = 1$ or 2, $a_2 = -1$ and $a_1$ any positive integer $> 1$ uses exactly the same argument. The special cases $\lambda(1; 1, -1) = 1$ and $\lambda(2; 1, -1) = 2$ are easily checked.

The final case is $k = 1$ or 2, $a_2 > 0$. Write $n = A_1A_2m$ and apply Theorem 1 with $d_1 = A_1$. Then $a_1d_1 + a_2d_2 \equiv 0 \pmod{k}$ and is clearly positive. The other conditions of the theorem are satisfied since $(A_1A_1 + a_2A_2, a_1a_2) = 1$, $(A_2m, a_1) = 1$ and $(A_1m, a_2) = 1$.

The following results apply to the case where the $x_i$ may be positive or negative, and can be proved similarly.

**Theorem 1'**. The equation

$$\frac{k}{n} = \frac{a_1}{x_1} + \frac{a_2}{x_2}$$

is solvable in integers $x_1, x_2$ such that $(a_1, x_1) = 1 = (a_2, x_2)$, if and only if there exist divisors (positive or negative) $d_1$ and $d_2$ of $n$ such that $a_1d_1 + a_2d_2 = kt$ for some positive integer $t$ such that $(a_1a_2, t) = 1$ and $(n/d_1, a_1) = (n/d_2, a_2) = 1$.

**Theorem 2'**. Let $(a_1, a_2) = (k, a_1a_2) = 1$, then $\lambda(k; a_1, a_2) = \infty$ unless

(i) $|a_1a_2| = 1$ and $k = 1, 2, 3, 4$ or 6

or

(ii) both $a_1$ and $a_2$ are $\pm 1$ and have the property that all primes dividing $a_1$ or $a_2 \equiv \pm 1 \pmod{k}$.

In these cases $\lambda = 0$ except that if $|a_1a_2| = 1$, $\lambda(3; a_1, a_2) = \lambda(4; a_1, a_2) = 1$ and $\lambda(6; a_1, a_2) = 2$.

Case (i) in the above theorem was previously mentioned in [6].
The Case $r = 3$. The solution of equation (1) with $r = 3$ and all of the $a_i = 1$ has received considerable attention. The finiteness of $\lambda(4; 1, 1, 1)$, $\lambda(5; 1, 1, 1)$, $\lambda(k; 1, 1, 1)$ and $\lambda'(k; 1, 1, 1)$ has been conjectured by Erdős, Strauss, Sierpinski and Schinzel. Although many people have considered the problem, it is not known if $\lambda(k; 1, 1, 1)$ is finite for any $k > 3$. A fairly complete list of references can be found in [1].

Efforts on the problem for $\lambda'$ have been a little more successful as Sierpinski [5], Sedláček [4], Palamà [3], and Stewart and Webb [6] have established that $\lambda'$ is finite for $k < 36$.

Although the conjectured values of $\lambda$ for small $k$ are small ($\lambda(4; 1, 1, 1) = 1$, $\lambda(5; 1, 1, 1) = 1$, $\lambda(7; 1, 1, 1) = 2$), some numerical evidence obtained by Webb [7] indicates that $\lambda$ increases rapidly with $k$. For example $\lambda(12; 1, 1, 1) \geq 12241$. In a private communication, Erdős noted that $\lambda(k; 1, 1, 1) > ck^{1+\varepsilon}$ for $c > 0$ and any $\varepsilon < \frac{1}{2}$, and conjectured that $\lambda(k; 1, 1, 1) > k^{s}$ for every positive integer $s$ and all $k$ sufficiently large.

In this section we prove this conjecture by establishing a slightly stronger inequality which holds for any $\lambda(k; a_1, a_2, a_3)$ and $\lambda'(k; a_1, a_2, a_3)$.

**Theorem 3.** There is a constant $c > 0$ such that

$$\lambda(k; a_1, a_2, a_3) > \exp(c \log k \log \log k)$$

for all $k$ sufficiently large.

**Proof.** Let $E = \exp (c \log k \log \log k)$. We will show that there exist primes $p$ in the interval $E \leq p \leq 2E$ for which the equation

$$\frac{k}{p} = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} \quad (a_i, x_i) = 1$$

has no solutions in positive $x_i$.

Without loss of generality we may suppose that $a_3/x_3$ is the largest of the three fractions $a_i/x_i$. This implies $k/3p \leq a_3/x_3$ and so $x_3 \leq 6a_3E/k$. Hence, there are at most $O(E/k)$ values of $x_3$ for which (5) is solvable for any $E \leq p \leq 2E$.

We now fix $x_3$ and bound the number of $p$ for which (5) is solvable with the given $x_3$.

$$\frac{k}{p} - \frac{a_3}{x_3} = \frac{kx_3 - pa_3}{px_3} = \frac{a_1}{x_1} + \frac{a_2}{x_2}.$$  

We note that $p > k$ and $p > x_3$ so $(p, k) = (p, x_3) = 1$. Also, $(kx_3 - a_3p, px_3) = 1$.

From (6) we see that $p \mid x_1x_2$.

Case I. Suppose $p \mid x_1$ and $p \mid x_2$. Then by Theorem 1 there exist $d_1$ and $d_2$ which divide $x_2$ such that $kx_3 - pa_3\mid a_1d_1 + a_2d_2$. We know $d_i \mid px_3$, but the condition $p \mid x_i$ implies $p \nmid d_i$. There are $d^2(x_3)$ choices for $d_1$ and $d_2$ and at most $d(a_1d_1 + a_2d_2)$ choices for $p$ given a particular $d_1$ and $d_2$. ($d(m)$ denotes the number of
Thus, there are at most \( d^2(x_3)d(a_1d_1 + a_2d_2) \leq f^3(E) \) values of \( p \) for which (6) is solvable, where \( f(n) \) is the maximum value of \( d(k) \) for all \( k \leq n \).

Case II. Suppose \( p \) divides only one of the integers \( x_1 \) and \( x_2 \). Say \( x_1 = py_1 \) and \( (x_2, p) = 1 \). Then

\[
\frac{kx_3 - pa_3}{px_3} - \frac{a_1}{py_1} = \frac{y_1(kx_3 - pa_3) - x_3a_1}{px_3y_1} = a_2,
\]

which implies \( p \mid y_1kx_3 - x_3a_1 \) and so \( p \mid y_1k - a_1 \). By Theorem 1, \( x_1 = px_3(a_1d_1 + a_2d_2)/kd_3 \) where \( d_1 \mid x_3p \) which implies \( y_1 < E^3/k^2 \) and so \( y_1k - a_1 < E^3 \). Hence, there are at most \( d(y_1k - a_1) < f(E^3) \) values of \( p \) for which (6) solvable.

Now by [2, Theorem 317] \( f(n) = O(\exp(\log n/\log \log n)) \), and so both \( f^3(E) \) and \( f(E^3) \) are \( O(\exp(3c \log k)) \). Therefore, the total number of primes \( p, E \leq p \leq 2E \) for which (5) is solvable, is \( O(\exp(3c \log k)E/k) \). However, there are at least \( E/\log^2 k \) primes between \( E \) and \( 2E \), and hence picking \( c < 1/3 \) we see that there must be some primes \( \geq E \) for which (5) is unsolvable.

**Corollary.** There is a constant \( c > 0 \) such that

\[
\lambda'(k; a_1, a_2, a_3) > \exp(c \log k \log \log k)
\]

for all \( k \) sufficiently large.

**Proof.** By the above argument, it is clear that there exist primes \( p, E \leq p \leq 2E \) such that all eight equations

\[
\frac{k}{p} = \pm a_1 \pm a_2 \pm a_3
\]

\( x_1 \quad x_2 \quad x_3 \)

are unsolvable.

There are a number of related questions which are still open and require further study. Some obvious examples are:

1. Can the bound on \( \lambda(k; a_1, a_2, a_3) \) be improved?

2. Can similar bounds be obtained for \( \lambda(k; a_1, a_2, a_3, a_4) \) or more generally for \( \lambda(k; a_1, \ldots, a_r) \)? (One result along these lines is that \( k = o(\lambda(k)) \). This is obvious from Lemma 1 of [6].)

**References**


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Author's address: Washington State University, Pullman, Washington, U.S.A.