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GENERALIZATION OF ONE BAER'S THEOREM FOR NETS

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As is well-known, R. BAER has proved in [1] that a projective plane is \((P, l)\)-desarguesian for a point \(P\) and a line \(l\) if and only if it is \((P, l)\)-transitive ([1], Theorem 6.2). In the present Note I shall generalize this Baer's theorem for nets of degree \(\geq 4\) provided \(P\) is a singular point and \(l\) the line of singular points.

After finishing the first version of this Note I got acquainted with the book [2] where an analogous problem for finite nets (of degree \(\geq 3\)) is considered in Chap. 4. Whereas I restricted myself to the configurative approach, [2] uses above all the algebraic (coordinatizing) methods and the case of degree 3 is not excluded. Our results show that the excluding of 3-nets (where the situation is known: cf. [3], p. 51) leads to a certain simplification, namely that only the Desargues condition is essential while the Reidemeister condition is superfluous and that the hypothesis of semi-regularity of automorphisms (either no point is fixed or all points are fixed) can be omitted.

Finally I wish to remark that I investigated also the influence of various specializations of the minor Desargues condition with respect to a net of degree \(\geq 4\) onto coordinatizing algebras in the paper [4] stimulated by former results of V. D. BELOUSHOV.

A non-trivial net (briefly: net) is defined as a triplet \((\mathcal{P}, \mathcal{L}, (V_i)_{i \in I})\) where \(\mathcal{P}\) is a non-void set, \(\mathcal{L}\) a set of some at least two-element subsets of \(\mathcal{P}\), \(I\) is an index set with \(\# I \geq 3\) and \(i \mapsto V_i\) an injective mapping of \(I\) into \(\mathcal{P}\) such that the following conditions are satisfied:

(i) \(\{V_i \mid i \in I\} \in \mathcal{L}\),
(ii) \(\forall P \in \mathcal{P} \setminus \{V_i \mid i \in I\} \quad \forall i \in I \quad \exists! l \in \mathcal{L} \quad P, V_i \in l\),
(iii) \(\forall a, b \in \mathcal{L}; \ a \neq b \quad \#(a \cap b) = 1\),
(iv) \(\#(\mathcal{P} \setminus \{V_i \mid i \in I\}) \geq 2\).  

Elements of \(\mathcal{P}\) are called points, elements of \(\mathcal{L}\) lines, points \(V_i, i \in I\), are termed singular (but here it will be more convenient to term them improper); also the line

\(^1\) If (iv) is changed to \(\#(\mathcal{P} \setminus \{V_i \mid i \in I\}) \leq 1\) then a trivial net arises.
\[
\{ V_i \mid i \in I \} \text{ will be termed improper whereas the remaining points and lines will be denoted as proper. The cardinality of } I \text{ is said to be the degree of the net.}
\]

By \( A_1, \ldots, A_n \) we write the fact that points \( A_1, \ldots, A_n \) lie on the same line. If \( A, B \) are distinct points then \( \# \{ l \in \mathcal{L} \mid A, B \in l \} = 0 \) or \( =1 \); in the latter case the only line through \( A, B \) will be designated by \( AB \). If \( a, b \), are distinct lines, then \( \# (a \cap b) = 1 \); the only common point of \( a, b \) will be designated by \( a \cap b \).

A quadruplet \((P, Q, R, S)\) is called a parallelogram if \( P, Q, R, S \) are proper points such that \( P, Q, V, R, S, \bar{V}, Q, R, W, P, S, W \) hold for suitable improper points \( V \neq W \). A triplet \((A, B, C)\) is called a triangle if \( A, B, C \) are proper points such that either \( A, B, C \) does not hold or \( A, B, C \) holds but \( A, B, C \) are not mutually distinct.

Now let \( \mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I}) \) be a net and let \( \alpha, \beta, \gamma \) be mutually distinct indices. Then the Reidemeister condition of type \((\alpha, \beta, \gamma)\) in \( \mathcal{N} \) is defined as the following implication: If \((P, Q, R, S), (P, Q, Q', P'), (Q, Q', R', R), (P, P', S', S)\) are parallelograms in \( \mathcal{N} \) such that \( P, S, V_\alpha, P, Q, V_\beta, P, P', V_\gamma \) hold for suitable improper points \( P, S, V_\alpha, P, Q, V_\beta, P, P', V_\gamma \) then also \((P', Q', R', S')\) is a parallelogram.

Let \( \mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I}) \) be a net of degree \( \geq 4 \) and \( \delta \) an index. Then the Desargues condition of type \((\delta)\) in \( \mathcal{N} \) is defined as the following implication: If \((A, B, C), (A', B', C)\) are triangles in \( \mathcal{N} \), if \((A, B, B', A'), (A, C, C', A')\) are parallelograms and if \( A, A', B, C \) is true\(^2\)) then \((B, C, C', B')\) is a parallelogram, too, or \( B, C, V_\delta \).

**Lemma 1.** Let \( \mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I}) \) be a net of degree \( \geq 4 \) and \( \delta \) an index. If \( \mathcal{N} \) satisfies the Desargues condition of type \((\delta)\) then \( \mathcal{N} \) satisfies also the Reidemeister condition of type \((\delta, \zeta, \eta)\) for all \( \zeta, \eta \) such that \( \delta, \zeta, \eta \) are mutually distinct.

**Proof.** Let the points \( P, Q, R, S, P', Q', R', S' \) satisfy the assumptions of the Reidemeister condition of type \((\delta, \zeta, \eta)\) in \( \mathcal{N} \) for arbitrarily chosen \( \zeta, \eta \). Choose another index \( \xi \neq \delta, \zeta, \eta \) which is possible since \( \mathcal{N} \) has degree at least 4. Then the points \( P, P'V_\xi \cap PV_\zeta, P', S, (P'V_\xi \cap PV_\zeta) V_\delta \cap SV_\zeta, S' \) satisfy the assumptions of the Desargues condition of type \((\delta)\) in \( \mathcal{N} \) so that \((P'V_\xi \cap PV_\zeta) V_\delta \cap SV_\zeta, S', V_\xi \) hold. Further, consider the points \( P, Q, PV_\xi \cap QV_\eta, R, SV_\xi \cap (PV_\xi \cap QV_\eta) V_\delta \). These points satisfy the assumptions of the Desargues condition of type \((\delta)\) in \( \mathcal{N} \), too, so that \( R, SV_\xi \cap (PV_\xi \cap QV_\eta) V_\delta \). Consequently \( R', SV_\xi \cap (PV_\xi \cap QV_\eta) V_\delta \). Finally, also the points \( PV_\xi \cap QV_\eta, P'V_\xi \cap PV_\zeta, Q', SV_\xi \cap (PV_\xi \cap QV_\eta) V_\delta, (P'V_\xi \cap PV_\zeta) V_\delta \cap SV_\zeta, R \) satisfy the assumptions of the Desargues condition of type \((\delta)\) in \( \mathcal{N} \) so that \((P'V_\xi \cap PV_\zeta) V_\delta \cap SV_\zeta, R', V_\xi \). The conclusions of the first and last application of the Desargues condition of type \((\delta)\) in \( \mathcal{N} \) imply \( S', R', V_\xi \). \( \blacksquare \)

\(^2\) We shall also say more briefly that points \( P, Q, R, S, P', Q', R', S' \) (in this arrangement) satisfy the assumptions of the Reidemeister condition of type \((\alpha, \beta, \gamma)\) in \( \mathcal{N} \).

\(^3\) We shall also say more briefly that points \( A, B, C, A', B', C' \) (in this arrangement) satisfy the assumptions of the Desargues condition of type \((\delta)\) in \( \mathcal{N} \).
Lemma 2. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I})$ be a net of degree $\geq 4$ satisfying the Desargues condition of type $\delta$ for some $\delta$. If $(1, 2, 2', 1'), (1, 3, 3', 1'), (2, 4, 4', 2')$ are parallelograms in $\mathcal{N}$ with $1, 1', V_0, 3, 4$ and with $3, 4, V_0$ then $(3, 4, 4', 3')$ is a parallelogram.

Proof. Let the points $1, 2, 3, 4, 1', 2', 3', 4'$ satisfy the assumptions of Lemma 2. If $1, 2, 3, 4$ then $(3, 4, 4', 3')$ is trivially a parallelogram. So let $1, 2, 3, 4$ be not true. Further let $(1, 2, 3, 4)$ be a parallelogram. Consider the points $1, 2, 3, 4, 1', 2'$. These points satisfy the assumptions of the Reidemeister condition of type $(8)$ for suitable $\xi, \eta$. By Lemma 1 this Reidemeister condition is valid in $\mathcal{N}$ so that $(3, 4, 4', 3')$ is a parallelogram as required. Now let $1, 2, 3, 4$ be not true and let $(1, 2, 3, 4)$ be not a parallelogram. Then for at least one of the pairs $(1, 2), (3, 4)$ there is a proper point $5$ such that $1, 2, 5, 3, 4$ or $1, 3, 5, 2, 4$, respectively. Let us consider the case $1, 2, 5, 3, 4$: If $a$ is the line through $1, 2, 5$ and $b$ the line through $3, 4, 5$ then $a \neq b$. Now $(1, 3, 5)$ and $(2, 4, 5)$ are necessarily triangles. Let $5'$ be such that $(1, 5, 5', V)$ is a parallelogram. Moreover, the points $1, 3, 5, 1', 3', 5'$ as well as $2, 4, 5, 2', 4', 5'$ satisfy the assumptions of the Desargues condition of type $(3)$ in $\mathcal{N}$ so that $3', 4', 5'$ lie on the line which possesses the same improper point as $b$. But then $(3, 4, 4', 3')$ is a parallelogram. The case $2, 4, 5, 2', 4', 5'$ can be dealt with similarly.

By an automorphism of a net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I})$ we mean a permutation $\pi$ of $\mathcal{P}$ such that every singular point is fixed under $\pi$ and $\{X^\pi \mid X \in \mathcal{L}\}$ is contained in a line of $\mathcal{N}$ for every $l \in \mathcal{L}$. For such a $\pi$ it follows $\{X^\pi \mid X \in l\}, \{X^{\pi^{-1}} \mid X \in l\} \in \mathcal{L}$ for all $l \in \mathcal{L}$. Thus $\pi$ induces a permutation $\pi$ of $\mathcal{L}$ with $l^\pi := \{X^\pi \mid X \in l\}$ for all $l \in \mathcal{L}$. If $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I})$ is a net and $\alpha$ an index then an $\alpha$-automorphism of $\mathcal{N}$ is an automorphism $\pi$ of $\mathcal{N}$ such that $l^\pi = l$ for every $l \in \mathcal{L}$ through $V_\alpha$. If moreover for any two proper points $A, A'$ with $A, A', V_\alpha$ there exists an $\alpha$-automorphism with $A \mapsto A'$ then $\mathcal{N}$ is said to be $\alpha$-transitive. It can be shown that $\mathcal{N}$ is $\alpha$-transitive if there is a proper line $l_0$ through $V_\alpha$ such that for any two proper points $A, A'$ on $l_0$ there exists an $\alpha$-automorphism with $A \mapsto A'$.

Theorem. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (V_i)_{i \in I})$ be a net of degree $\geq 4$ and $\delta$ an index. Then $\mathcal{N}$ satisfies the Desargues condition of type $\delta$ if and only if it is $\delta$-transitive.

Proof. a) Let $\mathcal{N}$ be $\delta$-transitive and let the points $A, B, C, A', B', C'$ satisfy the assumptions of the Desargues condition of type $\delta$ in $\mathcal{N}$. If $A, B, C$ are not mutually different then $(B, C, C', B')$ is trivially a parallelogram. If $A, B, C$ are mutually distinct then use a $\delta$-automorphism $\pi$ with $A^\pi = A'$. Then $(AB)^\pi = A'B', (AC)^\pi = A'C'$, $(BV_\delta)^\pi = BV_\delta, (CV_\delta)^\pi = CV_\delta$, so that $C' = (AC \cap CV_\delta)^\pi = A'C' \cap C'V_\delta = C'$, $B' = (AB \cap BV_\delta)^\pi = (AB)^\pi \cap (BV_\delta)^\pi = A'B' \cap BV_\delta = B'$. Therefore $(BC)^\pi = B'C'$ and since $\pi$ is a net automorphism, $BC$ and $B'C'$ must have the same improper point. Consequently $(B, C, C', B')$ is a parallelogram as claimed.
b) Let \( \mathcal{N} \) satisfy the Desargues condition of type \( (\delta) \). Start with an arbitrary couple \((A_0, A'_0)\) of proper points such that \( A_0, A'_0, V_\delta \) and define a mapping \( \pi_{A_0, A'_0} : \mathcal{P} \to \mathcal{P} \) as follows: 1) Every improper point will be fixed under \( \pi_{A_0, A'_0} \). 2) If \( X \) is a proper point, then let \( X' \) be a point for which an intermediating couple \((X_0, X^*_0)\) exists so that \( \left( A_0, X_0, X^*_0, A'_0 \right), \left( X_0, X, X', X^*_0 \right) \) are parallelograms. We shall show that \( X' \) is thereby determined in a unique way independently of \((X_0, X^*_0)\): Indeed, at least one intermediating couple \((X_0, X^*_0)\) exists because we can take arbitrary indices \( \alpha, \beta, \delta \) are mutually distinct and put \( X_0 := A_0 V_\alpha \cap X V_\beta, X^*_0 := A'_0 V_\alpha \cap X_0 V_\beta \) (consequently, \( X' := X^*_0 V_\beta \cap X V_\delta \)). Further, the independence of \( X' \) of the choice of \((X_0, X^*_0)\) is guaranteed immediately by Lemma 2. So we can declare \( X' \) to be the image of \( X \) under \( \pi_{A_0, A'_0} \).

Now it is clear that \( \pi_{A_0, A'_0} \) must be bijective (and thus a permutation of \( \mathcal{P} \)) as well as that \( \{ X^*_{A_0, A'_0} \mid X \in L \} = L \) for every line through \( V_\delta \). So it remains to show that also \( \{ X^*_{A_0, A'_0} \mid X \in L \} \subseteq L \) for every \( L \) not through \( V_\delta \): Let \( L \) be a line not through \( V_\delta \) (and therefore going through some \( V_\alpha, \alpha \neq \delta \)). Choose an index \( \beta \neq \alpha, \delta \) and put \( X_0 := A_0 V_\alpha \cap L, X'_0 := A_0 V_\alpha \cap X_0 V_\beta \). If \( X \) is an arbitrary proper point of \( L \) then construct \( X^*_{A_0, A'_0} \) by means of the intermediating couple \((X_0, X'_0)\). We see that if \( X \) runs over \( L \) then \( X^*_{A_0, A'_0} \) runs over \( X_0 V_\alpha \) i.e. \( \{ X^*_{A_0, A'_0} \mid X \in L \} \subseteq L \) as required.

\[ \text{References} \]


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