Milan Koman
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NEW UPPER BOUNDS FOR THE CROSSING NUMBER OF $K_n$
ON THE KLEIN BOTTLE

MILAN KOMAN, Praha
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1. INTRODUCTION

The crossing number of a graph $G$ for an orientable as well as for a nonorientable surface with the genus $g$ is defined as the least possible number of crossings in a drawing of $G$ on the mentioned surface. For an orientable surface we use the notation $cr_g^*(G)$, for the nonorientable one $cr_g(G)$, $g = 1, 2, 3, \ldots$.

The crossing numbers are most frequently investigated for the complete graphs $K_n$ and the complete bipartite graphs $K_{m,n}$. Most papers deal with the plane. The other surfaces mentioned are the torus, the projective plane and the Klein bottle.

We shall give briefly the most important results in the chronological order.

1. ZARANKIEWICZ 1954 [1] and URBANIK 1955 [2]:

$$cr_g^*(K_{m,n}) \leq \left(\frac{m}{2}\right)\left(\frac{m-1}{2}\right)\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right).$$

2. GUY 1960 [3], HARARY and HILL 1962 [4], BLAŽEK and KOMAN 1963 [5], SAATY 1964 [6]:

$$cr_g^*(K_n) \leq \frac{1}{4}\left(\frac{n}{2}\right)\left(\frac{n-1}{2}\right)\left(\frac{n-2}{2}\right)\left(\frac{n-3}{2}\right).$$

3. GUY, JENKYNs and SCHAER 1967 [7]:

$$cr_1^*(K_n) \leq \frac{59}{216} \left(\frac{n-1}{4}\right), \quad n \geq 10.$$

4. BLAŽEK and KOMAN 1967 [8, 9], HARBOURLTH 1971 [10]:

$$cr_g^*(K_{n_1,n_2,\ldots,n_u}) \leq \sum_{i} a_i b_i A_i B_i - \sum_{i<j} a_i b_i a_j b_j +$$

$$+ \sum_{r<s<i<u} (a_i a_j a_r a_u + a_i a_j c_r c_u + a_r c_s c_r c_u + c_r c_s a_r a_u + c_r a_i a_r c_u),$$

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where
\[ a_i = \left\lfloor \frac{n_i}{2} \right\rfloor, \quad b_i = \left\lfloor \frac{n_i - 1}{2} \right\rfloor, \quad c_i = \left\lceil \frac{n_i + 1}{2} \right\rceil, \]
\[ A_i = \left\lfloor \frac{(\sum n_j - n_i)}{2} \right\rfloor, \quad B_i = \left\lceil \frac{(\sum n_j - n_i - 1)}{2} \right\rceil. \]

5. Guy and Jenkyns 1969 [11]:
\[ cr_1^*(K_{m,n}) \leq \frac{1}{6} \binom{m-1}{2} \binom{n-1}{2}, \quad m, n \geq 45. \]

6. Koman 1969 [12, 13]:
\[ cr_1(K_n) \leq \frac{39}{128} \binom{n-1}{4}, \quad n \geq 10; \]
\[ cr_2(K_n) \leq \frac{37}{128} \binom{n-1}{4}, \quad n \geq 10. \]

7. Guy and Hill 1973 [14]:
\[ cr_0^*(\overline{C}_n) \leq \frac{1}{64} (n-3)^2 (n-5)^2, \quad n \text{ odd}; \]
\[ \leq \frac{1}{64} n(n-4) (n-6)^2, \quad n \text{ even}, \]

where \( \overline{C}_n \) is the complement of a circuit of the length \( n \).

\[ cr_2(K_{m,n}) \leq \frac{1}{6} \binom{m-1}{2} \binom{n-1}{2} \]
for infinite integers \( m \) and \( n \). It is a hiatus that the inequality holds for all \( m, n \geq 45 \).

From the given survey it is seen among other that on the torus as well as on the Klein bottle the same upper bounds for the crossing numbers of the complete bipartite graphs \( K_{m,n} \) hold.

In this paper we shall show that for the complete graph \( K_n \) the known value
\[ \frac{59}{216} \binom{n-1}{4} \]
is the upper bound not only for the torus but also for the Klein bottle.
2. PRECISE VALUES AND BOUNDS FOR $n \leq 15$

The results for the Klein bottle and for the torus are given simultaneously (see [7, 12]):

$$cr_2(K_7) = 1, \quad cr_1^*(K_7) = 0,$$

$$cr_2(K_8) = 4, \quad cr_1^*(K_8) = 4,$$

$$cr_2(K_9) = 9, \quad cr_1^*(K_9) = 9.$$

For $n = 10$ in contradistinction to the torus, where the precise value $cr_1^*(K_{10})$ is known, only the upper and lower estimates are known for the Klein bottle:

$$22 \leq cr_2(K_{10}) \leq 24, \quad cr_1^*(K_{10}) = 23.$$

The upper bounds for the both surfaces follow from Figs. 1 and 2.

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Fig. 1 ($cr_2(K_{10}) \leq 24$)

Fig. 2 ($cr_1^*(K_{10}) \leq 23$)

Fig. 3
For $11 \leq n \leq 15$ we can give for the Klein bottle as well as for the torus the following estimates:

\[
\begin{align*}
35 & \leq cr_2(K_{11}) \leq 43, & 37 & \leq cr_1^*(K_{11}) \leq 42, \\
53 & \leq cr_2(K_{12}) \leq 72, & 56 & \leq cr_1^*(K_{12}) \leq 70, \\
77 & \leq cr_2(K_{13}) \leq 109, & 81 & \leq cr_1^*(K_{13}) \leq 105, \\
108 & \leq cr_2(K_{14}) \leq 161, & 114 & \leq cr_1^*(K_{14}) \leq 154, \\
148 & \leq cr_2(K_{15}) \leq 225, & 156 & \leq cr_1^*(K_{15}) \leq 225.
\end{align*}
\]

The inequality $cr_2(K_{11}) \leq 43$ gives Fig. 3. The upper bound is here lower by 1 than that given in [12]. The inequality $cr_2(K_{15}) \leq 225$ is a consequence of the construction which will be presented in Part 3. For $n = 15$, the upper bound for $cr_2(K_{15})$ decreases from 239 (see [12]) to 225.

### 3. Bounds for $n \geq 16$

The drawing of the graphs $K_n$, $n \geq 3$ on the Klein bottle, which gives the upper bound for the crossing number $cr_2(K_n)$, arises by a simple modification of an analogous Guy-Jenkyns's drawing on the torus [17]. For $n = 12$ the both drawings are obvious from Figs. 4 and 5.

For other $n$'s the drawings for $K_n$ are generalizations of Figs. 4, 5. All $n$ vertices are divided into three approximately equal parts: white, black and grey. We dislocate the vertices according to Figs. 6, 7. After sticking the appropriate pairs of vertices we obtain in the first case the Klein bottle, in the second the torus.
On the constructed surface we join all the vertices having the same colour with the identically coloured circuit. In this way three disjoined circuits arise: white, black and grey, which form already a crossingfree subdrawing of the sought drawing. The edges joining two vertices with different colours are determined by these crossingfree circuits. The other edges are constructed according to the following rule.

First of all we label all the vertices with cyclical orders on each of the crossingfree coloured circuits with the integers

\[ 1, 2, 3, \ldots, n_i \]

\((n_i\) is the length of the white, black and grey circuit \(C_i\) respectively). We join two identically coloured vertices \(u, v\) coinciding with a crossingfree circuit \(C_i\) (with length \(n_i\)) on one side iff the edge \(uv\) is parallel to some of the edges

\[ 1w, \ w = 2, 3, \ldots, \lfloor n_i/2 \rfloor. \]

Other remaining joins of the vertices belonging to \(C_i\) are constructed on the opposite side of the circuit \(C_i\). Two edges \(uv, wt\), whose endpoints belong to the same crossingfree circuits \(C_i\) of length \(n_i\) are called parallel iff

\[ u + v \equiv w + t \pmod{n_i}, \]

holds.

The just obtained drawing of the graph \(K_n\) on the Klein bottle as well as that on the torus contains three subdrawings of complete graphs generated by white, black or grey vertices respectively. These subdrawings are homeomorphical with the drawing given in [5] by means of the construction \(B\).

From the table and from [12, 13] we obtain the inequalities

\[ \frac{1}{14} \binom{n}{4} \leq cr_2(K_n) \leq \frac{59}{216} \binom{n-1}{4}. \]
As in [7] we obtain the following upper bounds:

<table>
<thead>
<tr>
<th>$n =$</th>
<th>Common upper bound for $cr_2(K_n)$ and $cr_1(K_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6k = 3u$</td>
<td>$u(u - 2)(59u^2 - 98u + 24)/64 = n(n - 6)(59n^2 - 294n + 216)/5184$</td>
</tr>
<tr>
<td>$6k + 1 = 3u + 1$</td>
<td>$u(177u^3 - 412u^2 + 180u + 64)/192 =$</td>
</tr>
<tr>
<td></td>
<td>$(u - 1) (59n^3 - 589n^2 + 1541n - 435)/5184$</td>
</tr>
<tr>
<td>$6k + 2 = 3u - 1$</td>
<td>$(u - 1) (177u^3 - 707u^2 + 727u - 117)/192 =$</td>
</tr>
<tr>
<td></td>
<td>$(u - 2) (59n^3 - 530n^2 + 944n + 480)/5184$</td>
</tr>
<tr>
<td>$6k + 3 = 3u$</td>
<td>$(u - 1) (59u^3 - 157u^2 + 45u - 27)/64 =$</td>
</tr>
<tr>
<td></td>
<td>$(n - 3) (59n^3 - 471n^2 + 405n - 243)/5184$</td>
</tr>
<tr>
<td>$6k + 4 = 3u + 1$</td>
<td>$(u - 1) (177u^3 - 235u^2 - 97u + 27)/192 =$</td>
</tr>
<tr>
<td></td>
<td>$(n - 4) (59n^3 - 412n^2 + 356n + 240)/5184$</td>
</tr>
<tr>
<td>$6k + 5 = 3u - 1$</td>
<td>$(u - 2) (177u^3 - 530u^2 + 416u - 96)/192 =$</td>
</tr>
<tr>
<td></td>
<td>$(n - 5) (59n^3 - 353n^2 + 365n - 87)/5184$</td>
</tr>
</tbody>
</table>

To compare the upper bounds for the same graph (class of graphs) but for different surfaces it is useful to investigate the coefficients by the leading members:

<table>
<thead>
<tr>
<th></th>
<th>$K_n$</th>
<th>$K_m,n$</th>
<th>$\bar{c}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclidean plane</td>
<td>$\frac{1}{64} = 0.0156 \ldots$</td>
<td>$\frac{1}{16} = 0.0625 \ldots$</td>
<td>$\frac{1}{64} = 0.0156 \ldots$</td>
</tr>
<tr>
<td>Projective plane</td>
<td>$\frac{1}{1024} = 0.00126 \ldots$</td>
<td>$\frac{1}{2} \ldots$</td>
<td>$\frac{1}{16} \ldots$</td>
</tr>
<tr>
<td>Torus</td>
<td>$\frac{5}{5184} = 0.00113 \ldots$</td>
<td>$\frac{1}{4} = 0.0416 \ldots$</td>
<td>$\frac{1}{2} = 0.0416 \ldots$</td>
</tr>
<tr>
<td>Klein bottle</td>
<td>$\frac{5}{5184} = 0.00113 \ldots$</td>
<td>$\frac{1}{2} = 0.0416 \ldots$</td>
<td>$\frac{1}{2} = 0.0416 \ldots$</td>
</tr>
</tbody>
</table>

References


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Author's address: 116 39 Praha 1, Rettigové 4 (Katedra matematiky a fyziky pedagogické fakulty UK).