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Mirko Horňák

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Časopis pro pěstování matematiky, roč. 103 (1978), Praha

A THEOREM ON NON-EXISTENCE OF A CERTAIN TYPE OF NEARLY REGULAR CELL-DECOMPOSITIONS OF THE SPHERE

MIRKO HORŇÁK, Košice

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1. Introduction. In GRÜNBAUM [2] and MALKEVITCH [4] (cf. HORŇÁK-JUCOVIČ [3]) the following kind of maps on the sphere is investigated: The number of edges of every face of the map is a multiple of k (the faces are *multi-k-gonal*), the valency of its every vertex is a multiple of m (the vertices are *multi-m-valent*) with the exception of at most two vertices or faces (*exceptional* cells), m, k are integers greater than 1. CROWE [1] studies such maps with two exceptional faces with a prescribed *distance* (i.e., the length of the shortest path – in the sense of the graph theory – between a vertex of one exceptional face and a vertex of the other one). Here we present a result of this kind for $m = 3$ and $k = 5$. We are dealing with classes of cell-complexes decomposing the sphere in which all vertices are multi-3-valent and all faces are multi-5-gonal with the exception of a) one face or b) two adjacent faces or c) two vertices whose distance is 3. They are denoted $M(3, 5; 0, 1; 0)$, $M(3, 5; 0, 2, 0, \bar{0})$ and $M(3, 5; 2, 0; 0, 3)$, respectively (cf. Horňák-Jucovič [3]). The aim of the present paper is to prove emptiness of the class $M(3, 5; 2, 0; 0, 3)$ on the base of emptiness of the other two classes (proved by Malkevitch [4]).

2. Theorem. *The class* $M(3, 5; 2, 0; 0, 3)$ *is empty.*

Proof. Suppose that $P = (u_1, u_1w_1, w_1, w_1w_2, w_2, w_2u_2, u_2)$ is the shortest path joining the exceptional vertices u_1, u_2 of a complex $D_1 \in M(3, 5; 2, 0; 0, 3)$. For $3r$ edges with one end-vertex w_i, $i \in \{1, 2\}$, two possibilities can occur: a) $n \geqq 4$ edges lie on the same side of the path P. Then the above mentioned $3r$ edges can be denoted $e_1, e_2, \ldots, e_{3r}$ in the cyclic order around the vertex w_i so that the edges e_{n+1}, e_{3r} belong to the path P. b) At most three edges with one end-vertex w_i lie on every side of the path P. In this case the valency of w_i is either 3 or 6. In the case a) change the configuration around the vertex w_i as depicted in Fig. 1. In the new complex D_2 the valency of w_i is decreased by 3, the number of edges of the face X'_j ($j \in \{1, 2, 3\}$) is greater by 5 than that of the face X_j, new pentagons and 3-valent vertices arise, the path P has not changed and the distance between u_1 and u_2 remains 3. That is why the complex D_2 belongs to $M(3, 5; 2, 0; 0, 3)$. In this way the valency of the vertex w_i is successively decreased until the situation of the case b) is reached.

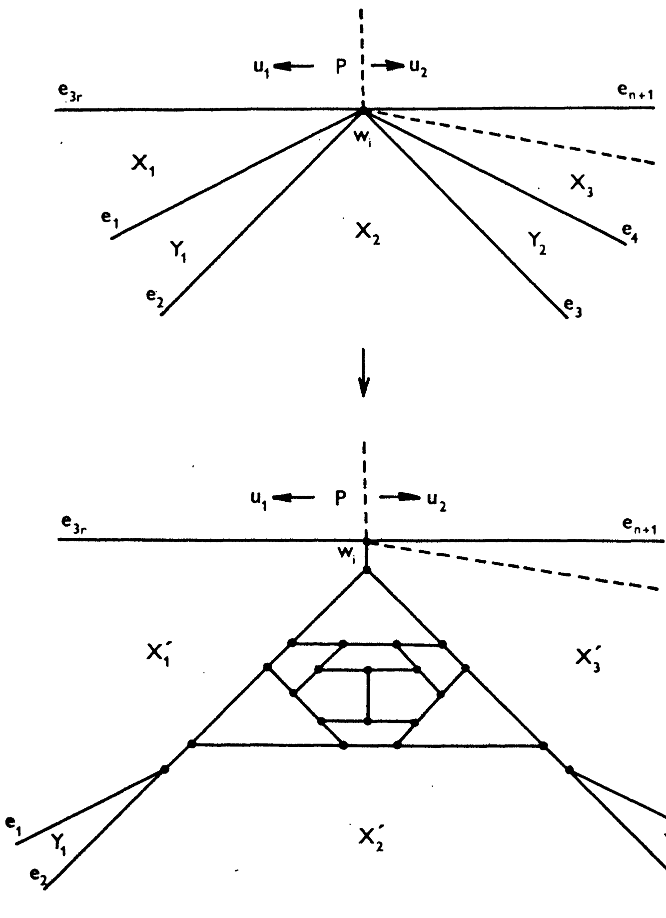

Fig. 1.

Similarly, if the valency of the exceptional vertex u_i is greater than 6, it is decreased by the above described procedure until it becomes 4 or 5; in this case the role of the edges e_1, e_2, e_3, e_4 is played by the four neighbouring edges with one end-vertex u_i which do not belong to the path P. These transformations lead to a final complex $D \in M(3, 5; 2, 0; 0, 3)$ with exceptional vertices u_1, u_2 of valencies 4 or 5 joined by the path P with inner vertices w_1, w_2 of valencies 3 or 6 (if the valency of w_i is 6, then at least one edge with one end-vertex w_i lies on both sides of P).

The complex D can be always depicted so that the upper side of the path P does not contain more edges with one end-vertex w_1 or w_2 than the lower side. As in D

the vertices w_1, w_2 are 3-valent or 6-valent, the upper side of P contains at most four edges; all possibilities are shown in Figs. 2a–2k (the dotted lines starting from the vertices w_1, w_2 indicate possible additional edges). It is suitable to distinguish three cases: a) u_1 and u_2 are 5-valent, b) the valency of u_1 differs from the valency of u_2; c) u_1 and u_2 are 4-valent.

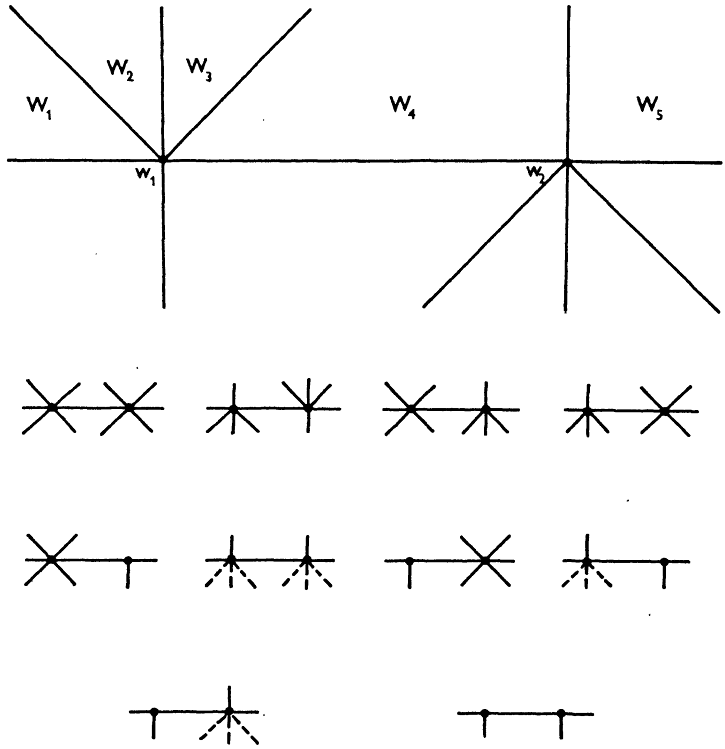

Fig. 2a–2k.

a) If u_1 and u_2 are 5-valent and the situation from Fig. 2a occurs for them, the faces W_i, $i \in \{1, 2, 3, 4, 5\}$, can be subdivided as marked in Fig. 3. In the new complex D' the vertices u_1, u_2 are no more exceptional (they are 6-valent), the faces W_i', $i \in \{1, 2, 4, 5\}$, are multi-5-gons, but the faces U_1, U_2 are not multi-5-gons (U_1 has the number of edges greater by 4 than the face W_3 of the complex D, U_2 is a 4-gon; in these faces the number 4 in Fig. 3 denotes their *type of exceptionality*: if X is a q-gonal exceptional face, its type of exceptionality is defined as the number $x \in \in \{1, 2, 3, 4\}$ such that $q \equiv x \pmod 5$) and they are the only exceptional cells of D'. Because of the adjacency of the faces U_1, U_2, the complex D' would be a member of the empty class $M(3, 5; 0, 2; 0, \bar{0})$ – a contradiction.

If the inner part of the path P looks like one of those in Figs. 2b, 2d, 2g or 2i, the faces of the upper side of P can be changed in accordance with Fig. 4, 5a, 6a or 7a, respectively (possible additional edges starting from w_1 or w_2, being unnecessary in the procedure of the construction, are omitted for simplicity in Figs. 6a and 7a). The resulting complex would be a member of $M(3, 5; 0, 1; 0)$ (Fig. 4) or $M(3, 5; 0, 2; 0, \bar{0})$ (Figs. 5a, 6a and 7a) in contradiction with the emptiness of these classes.

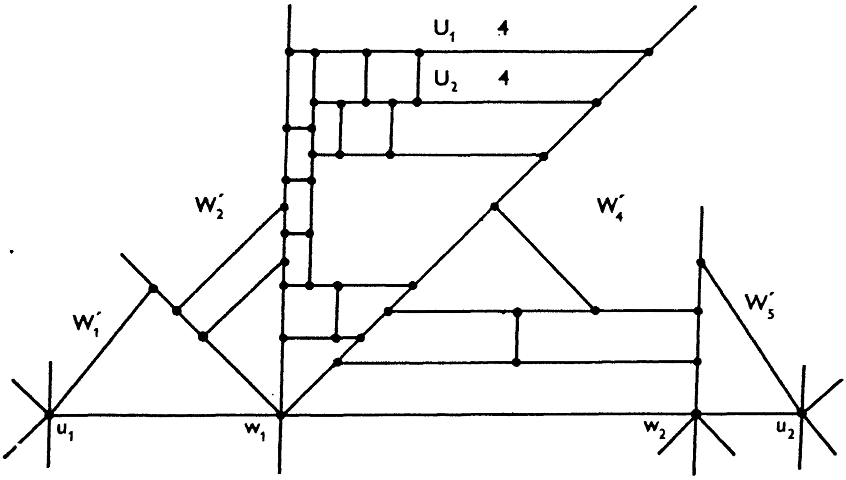

Fig. 3.

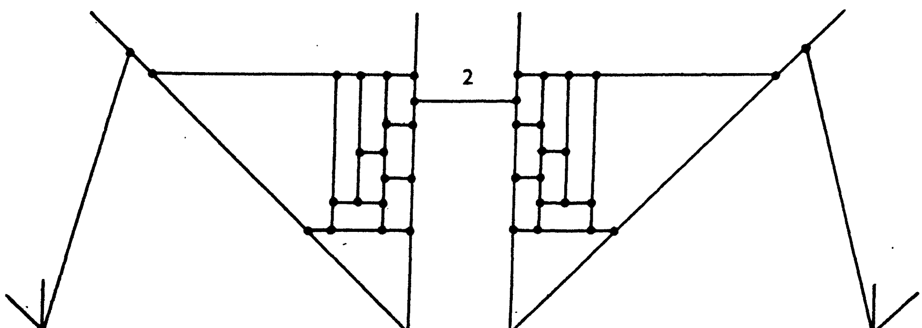

Fig. 4.

In the following part of the proof, three types of symmetry will be used — the symmetry with respect to the axis containing the centre of the edge w_1w_2 and perpendicular to w_1w_2, the symmetry with respect to the axis containing w_1w_2, and the composition of these symmetries, i.e., the symmetry with respect to the centre of w_1w_2; let us denote them *α-symmetry*, *β-symmetry* and *γ-symmetry*, respectively. (It is assumed, of course, that the edge w_1w_2 of the complex D is depicted as a line segment, in general, however, the above mentioned geometrical symmetries can be regarded only as symmetries in the combinatorial sense.)

It is clear that if a configuration C leads by a certain construction to a contradiction with the emptiness of the class $M(3, 5; 0, 1; 0)$ or $M(3, 5; 0, 2; 0, \bar{0})$, then the configuration σ-symmetrical ($\sigma \in \{\alpha, \beta, \gamma\}$) to C leads to the same type of contradiction

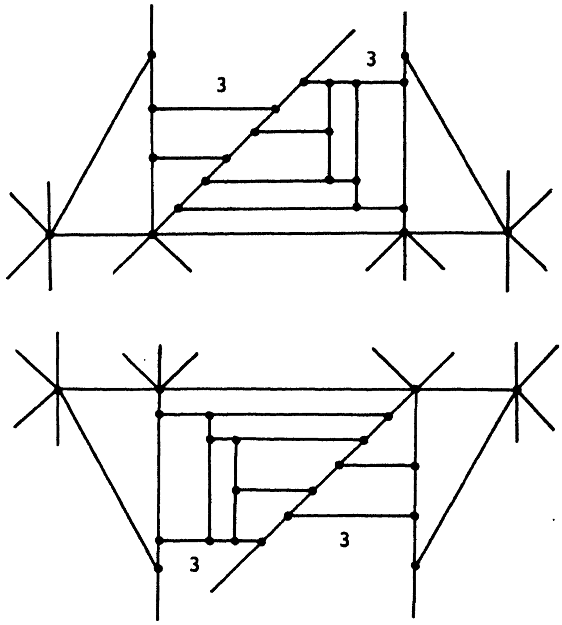

Fig. 5a, b.

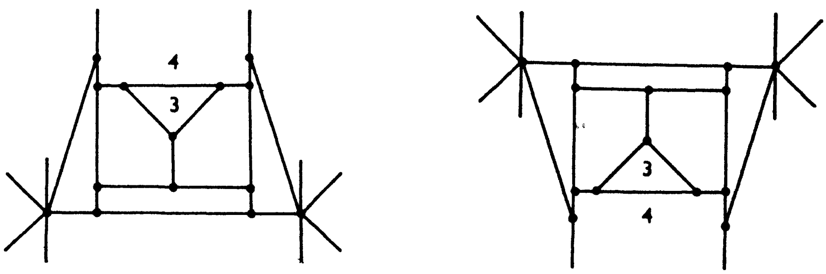

Fig. 6a, b.

by the construction which is σ-symmetrical to the one mentioned above. This fact is illustrated by the following examples: 2i and 2j are mutually α-symmetrical (as u_1 and u_2 are 5-valent, we may consider the whole path P without the loss of the α-symmetry) as well as Figs. 7a and 7b, the upper side of the configuration of Fig. 2g (2d) is β (γ)-symmetrical to the lower side of the configuration of Fig. 2k (2h), the same being true for Fig. 6a (5a) and Fig. 6b (5b).

As Fig. 2c (2e) is the image of Fig. 2a (2d) in the α-symmetry and the lower side of Fig. 2f is the image of the upper side of Fig. 2d in the β-symmetry, every possible shape of the path P with 5-valent exceptional vertices u_1, u_2 leads to a contradiction.

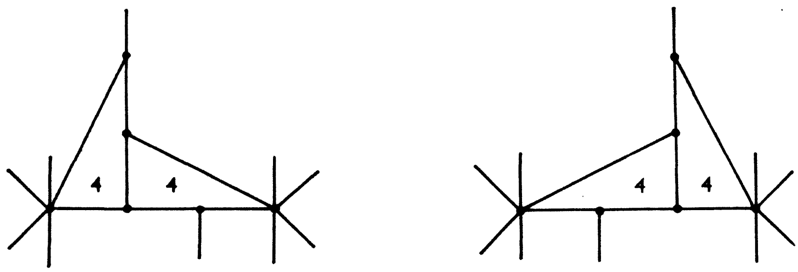

Fig. 7a, b.

b) and c) If at least one exceptional vertex is 4-valent, a contradiction with the emptiness of $M(3, 5; 0, 1; 0)$ or $M(3, 5; 0, 2; 0, \bar{0})$ can be reached quite analogously as in the preceding case by subdividing suitably the faces lying on one side of the path P. That is why the corresponding figures are omitted in this paper.

So if a complex with multi-3-valent vertices and multi-5-gonal faces has two exceptional vertices u_1, u_2 and no more exceptional cells, no path of length 3 joining u_1 and u_2 can exist; our Theorem is proved.

3. Remark. The assertion of Theorem does not hold only for cell-complexes, but for a much wider class of decompositions of the sphere, namely for maps whose countries are open discs.

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Author's address: 041 54 Košice, Komenského 14 (Katedra geometrie a algebry Prírodovedeckej fakulty UPJŠ).