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THE INSERTION OF REGULAR SETS IN POTENTIAL THEORY

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Introduction. In 1924, N. Wiener [8] proposed a new construction of the generalized solution of the Dirichlet problem for the Laplace equation. His method essentially uses the following fact: Any couple \((K, U)\) consisting of a compact set \(K\) and an open set \(U\) with \(K \subset U\) is admissible in the sense that there is a set \(V\) regular for the Dirichlet problem such that

\[ K \subset V \subset \overline{V} \subset U. \]

It is known that each couple \((K, U)\) is also admissible for a wide class of more general second order elliptic partial differential equations than the Laplace equation. In fact, this follows from a result of R.-M. Hervé [4] (Proposition 7.1) established in the context of Brelot harmonic spaces. A related question in the same context is also investigated in [6]. On the other hand, a similar result is no longer valid e.g. for the heat equation as observed by H. Bauer in [1], p. 147. Consequently, the original Wiener's procedure is not directly applicable. (Note that the Wiener type solution has recently been investigated in [7] in the frame work of the axiomatic potential theory.)

The aim of this paper is to study in terms of Bauer's axiomatics necessary and sufficient conditions guaranteeing that a couple \((K, U)\) is admissible. To this end, a special hull \(r(K)\) of \(K\) is introduced in a suitable way so that the main result reads then as follows: The couple \((K, U)\) is admissible, if and only if \(r(K) \subset U\). For the case of the heat equation, several characterizations of \(r(K)\) in terms of absorbent sets and balayage are given.

1. Terminology and notation. In what follows, \(X\) will denote a strong harmonic space in the sense of H. Bauer's axiomatics. For all notions we refer to [1]. For any set \(M\) we shall denote by \(M^*\), \(\text{int} M\) and \(\overline{M}\) its boundary, interior and closure, respectively.

Let \(U\) be an open subset of \(X\) and \(K\) a compact subset of \(U\). The couple \((K, U)\) is called admissible if there exists a regular set \(W\) such that \(K \subset W \subset \overline{W} \subset U\). For a
compact set $K \subset X$, we put
$$r(K) = \cap \{V; K \subset V \subset X; \text{V regular}\}.$$ If there is no regular set $V$ such that $K \subset V$, put $r(K) = X$.

2. **Lemma.** If $r(K) \neq X$, then
$$r(K) = \cap \{\overline{V}; K \subset V \subset X; \text{V regular}\} ;$$ in particular, $r(K)$ is compact.

**Proof.** According to Theorem 4.3.5 of [1] to each regular set $W$ such that $K \subset W$, there exists a regular set $W_0$ such that $K \subset W_0 \subset W$.  

3. **Theorem.** The following statements are equivalent:
   
   (i) a couple $(K, U)$ is admissible;
   (ii) $r(K) \neq X$, $r(K) \subset U$.

**Proof.** Implication (i) $\Rightarrow$ (ii) is obvious. Assume (ii) and let $W$ be a regular set such that $K \subset W$. We can limit ourselves to the case $\overline{W \cap (X \setminus U)} \neq 0$. Then $\overline{W \cap (X \setminus U)}$ is compact and $r(K) \cap (\overline{W \cap (X \setminus U)}) = 0$, i.e. $[\overline{W \cap (X \setminus U)}] \subset \subset [X \setminus \cap \{V; K \subset V, \text{V reg.}\}]$, thus
   $$\overline{W \cap (X \setminus U)} \subset \bigcup_{V_{\text{reg.}} \subset \overline{V}} (X \setminus V).$$

We can therefore choose regular sets $V_1, \ldots, V_n$ such that
   $$\overline{W \cap (X \setminus U)} \subset [X \setminus \bigcap_{i=1}^n V_i].$$

By Corollary 4.2.7 of [1], $\bigcap_{i=1}^n V_i$ is a regular set. Obviously,
   $$K \subset \bigcap_{i=1}^n V_i$$
and thus applying Theorem 4.3.5 of [1] we can find a regular set $V_0$,
   $$K \subset V_0 \subset \bigcap_{i=1}^n V_i .$$

Put $W_0 = V_0 \cap W$. Then $K \subset W_0$, $W_0$ is (according to Corollary 4.2.7 of [1] again) regular. Moreover, $W_0 \subset U$.

4. **Notation.** For $E \subset X$, let $A(E, X)$ be the smallest absorbent set in $X$ containing $E$. We shall write $A(x, X)$ instead of $A(\{x\}, X)$.
5. Lemma. The components of an absorbent set are absorbent sets.

Proof. For S connected, \( A(S, X) \) is always connected. (See Exercise 6.1.2 in [3].) Let \( B \) be a component of \( A \). Then \( A(B, X) \) is a connected absorbent set containing \( B \). Consequently, \( B = A(B, X) \) and \( B \) is absorbent.

In what follows, \( X \) will denote the harmonic space corresponding to the heat equation on a Euclidean space \( R^{n+1} (n \geq 1) \) (see [1], Standard-Beispiel 2, p. 20).

6. Notation. Given a compact set \( K \subset X \), the parabolic hull \( M_K \) of \( K \) is the union of \( K \) and the set of all \( x \in X \setminus K \) for which \( A(x, X \setminus K) \) is relatively compact. Denote by \( T_K \) the union of \( K \) and the set of all \( x \in X \setminus K \) for which there exists no absorbent set \( B \) in \( X \) such that \( 0 \neq B \subset A(x, X \setminus K) \).

Further put \( L_K = \{ x \in X ; R^*_n(x) = 1 \} \).

7. Theorem. For a compact subset \( K \subset X \),

\[ r(K) = M_K = T_K = L_K. \]

Thus, together with Theorem 3 we obtained a characterization of admissible couples \((K, U)\) in terms of the parabolic hull of \( K \).

The proof of this theorem will be divided into the following steps.

8. Proposition. Let \( Y \) be an open subset of \( X \) and \( A \) a closed set in \( Y \). Then the following assertions are equivalent:

(i) The set \( A \) is absorbent in the harmonic space \( Y \).

(ii) For each \( x \in A \) there exists a neighborhood \( U_x \) and an absorbent set \( B \) in \( X \) such that \( U_x \cap A = U_x \cap B \).

Proof. Suppose (i). For \( x \in \text{int} \ A \), choose a neighborhood \( U_x \) of \( x \) such that \( U_x \subset A \), and put \( B = X \). If \( x \in Y \) is a boundary point of \( A \), then we choose \( a > 0 \) in such a way that the set

\[ U_x = \{ y \in R^{n+1} ; \sum_{i=1}^{n} (y_i - x_i)^2 - (a + x_{n+1} - y_{n+1})^2 < 0 ; x_{n+1} - a < y_{n+1} < x_{n+1} + a \} \]

is contained in \( Y \). (The sets of this form will be called standard cones. Recall that each standard cone is a regular set — see [1], p. 21). For each \( y \in U_x \cap A(x, X) \), \( y \neq x \), there is a standard cone \( S \) such that \( x \in S \subset U_x \cap Y \cap S^* \). Then \( y \in \text{spt} \mu^S_x \), where \( \mu^S_x \) denotes the harmonic measure corresponding to \( x \) and the regular set \( S \) (see [1], p. 21). Obviously, \( \text{spt} \mu^S_x \subset A \) and hence

\[ U_x \cap A(x, X) \subset U_x \cap A. \]
Suppose now that there exists $z \in (U_x \cap A) \setminus A(x, X)$. The supports of harmonic measures $\mu^V_x$ corresponding to regular sets $V$, $V \subset U_x$ (consider e.g. standard cones) for which $z \in V$, cover the set $[A(z, X) \cap U_x] \setminus \{z\}$. Thus

$$x \in \text{int} [U_x \cap A(z, X)] \subset U_x \cap A,$$

which yields a contradiction with the assumption that $x$ is a boundary point of $A$.

So we obtain $U_x \cap A(x, X) = U_x \cap A$ and we can put $B = A(x, X)$.

Now suppose (ii). By [2] absorbent sets in $X$ are exactly those which are closed and finely open. It follows that there is a fine neighborhood $V_x$ of $x$, contained in $B$. Since $U_x \cap V_x$ is a fine neighborhood of $x$ contained in $A$, $A$ is finely open, and (using [2] again) $A$ is an absorbent set in $Y$.

9. Corollary. Let $Y$ be an open subset of $X$. For each component $Q$ of the boundary of an absorbent set in $Y$ there exists $c \in \mathbb{R}$ such that $Q \subset \{x \in X; x_{n+1} = c\}$.

10. Lemma. For a compact $K \subset X$, $M_K \subset r(K)$.

Proof. Assume that $K \neq 0$ and choose $x^0 \in M_K \setminus K$. The standard cones are regular, hence $r(K) \neq X$. Suppose that there is a regular neighborhood $V$ of $K$, such that $x^0 \notin V$. Putting

$$L = \{x \in X; x_i = x^0_i \text{ for all } 1 \leq i \leq n, x_{n+1} \leq x^0_{n+1}\},$$

there exists $y \in L$ such that

$$y_{n+1} = \sup \{x_{n+1}; x \in L \setminus A(x^0, X \setminus K)\}.$$

According to Proposition 8, $y_{n+1} < x^0_{n+1}$. Denote

$$L_0 = \{x \in L; x_{n+1} > y_{n+1}\}.$$

By Proposition 8, $y \notin A(x^0, X \setminus K)$. Simultaneously $y \in A(x^0, X \setminus K)$ and hence $y \in K$. It follows $L_0 \cap V^* \neq 0$ and using the fact that $L_0 \subset A(x^0, X \setminus K)$, we have

$$0 \neq L_0 \cap V^* \subset A(x^0, X \setminus K) \cap V^*.$$

Let $y^0 \in A(x^0, X \setminus K)$ be chosen such that

$$y^0_{n+1} = \min \{x_{n+1}; x \in A(x^0, X \setminus K) \cap V^*\}.$$

First, consider the case when $y^0$ is a boundary point of $A(x^0, X \setminus K)$ relatively to the set $X \setminus K$. Using Proposition 8, there is a neighborhood $U_{y^0}$ of $y^0$ such that

$$U_{y^0} \cap (X \setminus V) \subset \{x \in X; y^0_{n+1} \leq x_{n+1}\}.$$

It follows (cf. [1], Theorem 4.3.1. and p. 108) that $y^0$ is an irregular boundary point of $V$, which is a contradiction. Using a similar argument, $y^0$ cannot be in the
interior of $A(x^0, X\setminus K)$. Thus, $M_K \setminus K \subset V$ and since $V$ is an arbitrary regular set containing $K$, we have $M_K \setminus K \subset r(K)$. Obviously, $K \subset r(K)$.

The proof of the inclusion $r(K) \subset M_K$ will be more complicated.

11. Lemma. For a compact set $K$ in $X$, the set $\{x \in X; \hat{R}_t^K(x) = 1\}$ is bounded.

Proof. Obviously it is sufficient to prove that $\{x \in X; \hat{R}_t^K(x) = 1\}$ is bounded for

$$K = \{x \in X; |x| \leq a_i, i = 1, \ldots, n + 1\} \quad (a_i \geq 0) .$$

(a) If $y \in X$ is such that $y_{n+1} < -a_{n+1}$, then

$$\hat{R}_t^K(y) = R_t^K(y) = 0 .$$

We can take the superharmonic function (see [1], p. 34.)

$$u = \begin{cases} 0 & \text{on } A(y, X), \\ 1 & \text{on } X \setminus A(y, X). \end{cases}$$

(b) If $y \in X$ is such that $|y_i| \leq a_i$ for $i = 1, \ldots, n, y_{n+1} > a_{n+1}$ consider the set

$$D = \{x \in X \setminus K; |x| < a_i + 1 \quad \text{for} \quad i = 1, \ldots, n, |x_{n+1}| < |y_{n+1}| + 1\} .$$

Obviously, $y \in D$. Choose $z \in D$, $z_i = -a_i - \frac{1}{2}$. Using (a), $\hat{R}_t^K(z) = 0$. Applying the maximum principle for the heat equation (e.g. Theorem 2.3 in [5] — note that $\hat{R}_t^K$ is a harmonic function on $D, \hat{R}_t^K \leq 1$) we obtain $\hat{R}_t^K(y) < 1 .

(c) In the case that for $y \in X, y_{n+1} \geq -a_{n+1}$ and there exists $i$ ($i = 1, \ldots, n$) such that $|y_i| > a_i$ we can proceed analogously.

12. Notation. For a compact set $\emptyset \neq K \subset X$, we define a sequence $\{K_n\}$:

$$K_n = \{x \in X; \text{dist} (x, K) \leq 1/n\} .$$


Proof. Let $K \neq \emptyset$ and consider $x^0 \in X \setminus M_K$. The set $A(x^0, X \setminus K)$ is unbounded, thus using the preceding lemma and Proposition 8, there is $y \in \text{int} A(x^0, X \setminus K)$ such that $\hat{R}_t^K(y) < 1$. The function $1 - \hat{R}_t^K$ is harmonic on $X \setminus K$. By the Harnack inequality (see [1], Theorem 1.4.4) applied to $X \setminus K$ and to the Dirac measure at $x^0$ there is $\alpha \geq 0$ such that

$$0 < 1 - \hat{R}_t^K(y) \leq \alpha(1 - \hat{R}_t^K(x^0)) .$$

It follows that $\hat{R}_t^K(x^0) < 1$.

Thus we proved that $L_K \subset M_K$. Let $y^0 \in M_K \setminus K$, choose $n_0$ such that $y^0 \not\in K_{n_0}$. Let $n \geq n_0$ be a natural number. According to Proposition 8 we obtain that the
"parabolic boundary" (see [5] Chap. 3) of \( \text{int} A(y^0, X \setminus K) \) in \( X \) is contained in \( K \). Using the fact that \( \hat{R}^K_n(y) = 1 \) for all \( y \in K \) together with the minimum principle for superharmonic functions for the heat equation (see Theorem 2.1 in [5]), we have

\[
\inf \{ \hat{R}^K_n(y); y \in \text{int} A(y^0, X \setminus K) \} = 1.
\]

Since \( y^0 \notin K \), \( \hat{R}^K_n \) is continuous at \( y^0 \) (compare with Corollary 2.3.5 in [1]) and \( \hat{R}^K_n(y^0) = R^K_n(y^0) = 1 \). Now, applying the assertion of Appendix 3.2.1 of [1] we have

\[
R^K = \inf_{n \in \mathbb{N}} R^K_n,
\]

and hence \( R^K(y^0) = 1 \) (note that \( K_n \to K_{n_0} \) for \( n < n_0 \) and \( R^K_n \geq R^K_{n_0} \)). This means \( y^0 \in L_K \). Obviously, \( K \subseteq L_K \).

14. Remark. In the course of the preceding proof we used the equality

\[
R^K = \inf_{n \in \mathbb{N}} R^K_n.
\]

It is an easy consequence that

\[
\{ x \in X; R^K_1(x) = 1 \} = \bigcap_{n=1}^{\infty} \{ x \in X; R^K_n(x) = 1 \}.
\]

Obviously, \( \{ x \in X; R^K_1(x) = 1 \} \cup K = \{ x \in X; R^K(x) = 1 \} \), so that

\[
\bigcap_{n=1}^{\infty} \{ x \in X; R^K_n(x) = 1 \} = \bigcap_{n=1}^{\infty} \{ x \in X; \hat{R}^K_n(x) = 1 \}.
\]

15. Lemma. For a compact \( K \subset X \), \( r(K) \subset M_K \).

Proof. Assume that \( K \neq \emptyset \). Consider \( x^0 \notin M_K \). Using Lemma 13 and the preceding remark, there exists a natural number \( n \) such that \( \hat{R}^K_m(x^0) < 1 \) for all \( m \geq n \). Simultaneously,

\[
\inf_{x \in M_K} \hat{R}^K_m(x) = 1.
\]

The set \( M_K \) is a closed subset of the compact set \( r(K) \). Hence, using Proposition 3.1.2 of [3] there is a fundamental system of regular neighborhoods of \( M_K \) not containing the point \( x^0 \). Thus, \( x^0 \notin r(K) \).

16. Lemma. \( T_K = M_K \).

Proof. Suppose first that \( x \in M_K \setminus T_K \). If \( B \) is an absorbent set in \( X \) such that \( B \subset A(x, X \setminus K) \), then \( B \) is a compact absorbent set and hence (see [1], p. 31) must be empty. It follows that \( M_K \subset T_K \). Suppose now that the set \( A(x, X \setminus K) \) is unbounded. Let \( D \supset K \) be an \((n + 1)\)-dimensional cube in \( X \) such that its faces are
parallel to the coordinate axes. Choose \( x^0 \in A(x, X \setminus K) \cap (X \setminus D) \). Applying Proposition 8, there is \( y^0 \in A(x, X \setminus K) \) such that

\[
y^0_{n+1} < \min_{x \in D} x_{n+1}.
\]

Again by Proposition 8, \( B = A(y^0, X) \subset A(x, X \setminus K) \).

17. **Proposition.** Let \( E \) be a compact subset of \( X \). If \( E \) is convex (or more generally, if the set \{ \( x \in E \); \( x_{n+1} = c \) \} is convex for each \( c \in \mathbb{R} \)) then \( r(E) = r(E^*) = E \).

**Proof.** Consider \( x^0 \in X \setminus E \) and let \( P \) be an arbitrary line which contains \( x^0 \), \( P \subset \{ x \in X \}; x_{n+1} = x^0_{n+1} \}. \) Consider \( A(x^0, X \setminus E) \) and denote by \( P_1 \) the half-line starting from \( x^0 \) for which \( P_1 \cap E = \emptyset \). Then according to Proposition 8, \( P_1 \subset A(x^0, X \setminus E), \) i.e. \( A(x^0, X \setminus E) \) is unbounded. This means \( x^0 \notin r(E) \). Thus we have \( r(E) \subset E \). Obviously \( E \subset r(E) \). Analogously we can show that \( r(E^*) \subset E \). Further, if \( x^0 \in \text{int} E \), then \( \text{int} E \) is closed and open — hence also finely open — in \( X \setminus E^* \). By [2] \( \text{int} E \) is an absorbent set in \( X \setminus E^* \). Hence \( A(x^0, X \setminus E^*) \subset \text{int} E \), i.e. \( A(x^0, X \setminus E^*) \) is bounded and \( x^0 \in r(E^*) \). Simultaneously \( E^* \subset r(E^*) \) and this completes the proof.

**References**


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