Josef Dalík
Strong elements in lattices

Časopis pro pěstování matematiky, Vol. 104 (1979), No. 3, 243--247

Persistent URL: http://dml.cz/dmlcz/118019

Terms of use:

© Institute of Mathematics AS CR, 1979

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
STRONG ELEMENTS IN LATTICES

JOSEF DALÍK, Brno

(Received June 18, 1976)

0. INTRODUCTION

The concept of a strong element has appeared in [2]. There it is used for the lattice-theoretical characterization of certain properties, which were defined for elements of the alphabet of a formal language in the work [1].

In the present note we discuss the relationship between strong elements and those satisfying “the upper covering condition”. We study the structure of the system of all strong elements in a given lattice and give two characterization theorems for lattices in which all elements are strong.

In the following, the cardinality of a set $S$ is denoted by $\text{card} \ S$. For an element $a \in S$ and for an equivalence relation $\equiv$ on $S$ the symbol $[a]_{\equiv}$ denotes the set of all elements $b \in S$ with the property $a \equiv b$.

The lattice operations are denoted by $\lor, \land$ and the appropriate partial ordering by $\leq$. The symbols $<, <$ and $\parallel$ are used in the standard sense. The reader is expected to be familiar with the notions of an atom and of the greatest (the smallest) element in a lattice. For a subset $S$ of a lattice $L$ we denote by $\inf_L S$ the infimum of $S$ in $L$. If $i$ is the greatest element in $L$, we define $\inf_L \emptyset = i$. We take advantage of the lattice-theoretical duality principle.

1. ELEMENTS FULFILLING THE UPPER COVERING CONDITION

1.1. Definition. Let $L$ be a lattice. An element $a \in L$ is called strong (in $L$), if it holds

$$b < c, a \parallel c \Rightarrow a \lor b < a \lor c$$

for any elements $b, c \in L$. If $a$ is strong in the lattice dual to $L$, it is called dually strong (in $L$).

We denote by $S_L$ the set of all elements, which are strong in $L$. If each element of $L$ is strong (dually strong), we call $L$ a strong (dually strong) lattice.
1.2. Lemma. Let $a$ be an element of a lattice $L$. Then $a \in S_L$ if and only if

$$a \parallel b \Rightarrow b < a \lor b$$

for any element $b \in L$.

Proof. Suppose $a \parallel b$, $b \nless a \lor b$ for an element $b \in L$. Then there exists an element $c \in L$ with the property $b < c < a \lor b$. It is obvious that $a \parallel c$, $b < c$ and $a \lor c \leq a \lor b$. Thus $a \notin S_L$.

Now let $a \notin S_L$. Then it holds $b < c$, $a \parallel c$ and $a \lor b = a \lor c$ for some elements $b, c \in L$. Thus $a < a \lor b$, $b < a \lor b$ and then, clearly, $a \parallel b$. Simultaneously, $b < c < a \lor b$ and the above-mentioned implication does not hold.

1.3. Definition. An element $a$ of a lattice $L$ fulfils the upper covering condition (in $L$), if it holds

$$a \land b < a \Rightarrow b < a \lor b$$

for any element $b \in L$. If all elements of $L$ satisfy this condition, we say that $L$ fulfils the upper covering condition.

1.4. Corollary. Each strong element of a lattice $L$ fulfils the upper covering condition in $L$.

1.5. Lemma. Let a lattice $L$ have a smallest element. An atom $a \in L$ is strong if and only if it fulfils the upper covering condition.

Proof. The statement follows from 1.2 and 1.4.

1.6. Corollary. If the lattice $L$ fulfils the upper covering condition, then each atom in $L$ is strong.

2. THE SET OF ALL STRONG ELEMENTS IN A LATTICE

2.1. Lemma. In any lattice the infimum of an arbitrary set of strong elements is strong.

Proof. Let us put $a = \inf_L M$ for a lattice $L$ and for $M \subseteq S_L$. In case $M = \emptyset$, $a$ is the greatest element in $L$ and, clearly, $a$ is strong.

For $M \neq \emptyset$ suppose $a \notin S_L$. Then it holds $b < c$, $a \parallel c$ and $a \lor b = a \lor c$ for some elements $b, c \in L$. Since $b \nless a$, there exists an element $m \in M$ with the property $b \nless m$. Then $b \nless m \lor b$, because $b < c < a \lor b \leq m \lor b$ and, obviously, $m \parallel b$. By 1.2, we obtain a contradiction.
2.2. Lemma. Let $a, b$ be strong elements of a lattice $L$ such that $a \parallel b$. Then it holds

$$c < a \iff c < b$$

for any $c \in L$.

Proof. Suppose $c < a$ for an element $c \in L$. It follows that $b \lor c < b \lor a$ and, clearly, $b \not\leq c$. Let us admit $b \parallel c$. Then it holds $b < b \lor c < b \lor a$. This implies $a \notin S_L$ according to 1.2 and we have a contradiction. It remains $c < b$.

The converse implication follows by interchanging the symbols $a, b$ in the preceding consideration.

2.3. Theorem. Let $L$ be a complete lattice. Then $S_L$ is a complete strong lattice satisfying

$$\inf_L M = \inf_{S_L} M$$

for any set $M \subseteq S_L$.

Proof. Theorem 10 in [3] and 2.1 say that $S_L$ is a complete lattice satisfying the required equalities.

Suppose $a \in S_L$ and $b < c$, $a \parallel c$ for elements $b, c \in S_L$. Then $b < a$ by 2.2. It follows $a \lor b = a < a \lor c$ and $a$ is strong in $S_L$.

3. CHARACTERIZATIONS OF STRONG LATTICES

3.1. Theorem. A lattice is strong if and only if it does not contain a sublattice isomorphic to one of the lattices $L_5, L_6$, the graphic description of which is given in Fig. 1.

![Fig. 1.](image)

Proof. Let the lattice $L$ be not strong. Then there exist elements $a, b, c, d$ in $L$ so that $a \parallel c$, $b < c$ and $a \lor b = a \lor c = d$. Let us denote $e = a \land b$, $f = a \land c$. Obviously, $e \leq f$. If $e = f$, then the elements $a, b, c, d, e$ form the sublattice $L_5$ in $L$. If $e < f$, then $e \lor x = x$ for $x = a, b, c, d, e, f$ and $f \lor x = x$ for $x = a, c, d, f$. Clearly, $f \lor b \leq c$. 

245
In case \( f \lor b = c \) the elements \( a, b, c, d, e, f \) form the sublattice \( L_6 \) in \( L \). If \( f \lor b < c \), then the elements \( f \lor b, a, c, d, f \) form a sublattice in \( L \) isomorphic to \( L_5 \).

It is obvious that any sublattice of a strong lattice is isomorphic neither to \( L_5 \) nor to \( L_6 \).

3.2. **Corollary.** A lattice is strong if and only if it is dually strong.

**Proof.** The assertion follows from 3.1, the dual statement and from the fact that the lattices \( L_5 \) and \( L_6 \) are selfdual.

3.3. **Definition.** Let \( L \) be a lattice and let \( \text{id}_L \) denote the identity relation on \( L \). We put \( \|_r = \| \cup \text{id}_L \).

It is clear that the relation \( \|_r \) is reflexive and symmetric in any lattice. The following theorem characterizes lattices in which \( \|_r \) is an equivalence.

3.4. **Theorem.** Let \( L \) be a lattice. Then \( L \) is strong if and only if \( \|_r \) is a transitive relation on \( L \).

**Proof.** Let \( L \) be strong. Let \( a \|_r b, b \|_r c \) for elements \( a, b, c \in L \). In case \( a = b \) and/or \( b = c \) we have \( a \|_r c \). In the opposite case it holds \( a \| b, b \| c \). Suppose \( a \not\| c \). Then \( c < a \) or \( a < c \). However, \( c < a \), \( a \| b \) give \( c < b \) by 2.2 and \( a < c, a \| b \) imply \( b < c \) according to the dualization of 2.2. In both cases we have a contradiction.

Now let \( \|_r \) be transitive on \( L \). An arbitrary element \( a \in L \) is strong: Let \( a \| b \) for \( b \in L \). If there exists an element \( c \in L \) with the property \( b < c < a \lor b \), then \( a \| c \). The symmetry and transitivity of \( \|_r \) give \( b \| c \), which is a contradiction. Thus, \( b < a \lor b \) and \( a \) is strong by 1.2.

3.5. **Lemma.** Let \( L \) be a strong lattice and let us assume \( a < b \) for \( a, b \in L \). Then it holds \( a' < b' \) for any elements \( a' \in [a]_{\|_r}, b' \in [b]_{\|_r} \).

**Proof.** Suppose that \( a' \|_r a \) and \( b' \|_r b \) for \( a', b' \in L \). As \( \|_r \) is an equivalence and \( a \not\|_r b \), it holds \( a' \not\|_r b' \). That means \( a' < b' \) or \( b' < a' \). By 2.2, the validity of \( b' < a' \) gives \( b' < a \). It implies \( b' < b \), which is a contradiction. Thus, we have proved \( a' < b' \).

3.6. **Lemma.** Let \( a, b \) be elements of a strong lattice \( L \) with the property \( a < b \). Then

\[
\text{card } [a]_{\|_r} > 1 \Rightarrow \text{card } [b]_{\|_r} = 1.
\]

**Proof.** Let us suppose \( \text{card } [a]_{\|_r} > 1 \) and \( \text{card } [b]_{\|_r} > 1 \). Then there exist elements \( a', b' \in L \) such that \( a' \| a \) and \( b' \| b \). By 3.5, it holds \( a' < b, a' < b' \) and \( a < b' \). It follows \( a < a \lor a' \leq b \land b' < b \), which contradicts \( a < b \).
The statements 3.4, 3.5 and 3.6 give a clear description of the structure of strong lattices. Three examples of these lattices are given in Fig. 2.

References


Author's address: 662 37 Brno, Barvičova 85 (Vysoké učení technické).