Ladislav Nebeský
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ON THE EXISTENCE OF A 3-FACTOR IN THE FOURTH POWER OF A GRAPH

LADISLAV NEBESKÝ, Praha
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Let G be a graph in the sense of [1] or [2]. We denote by \( V(G) \) and \( E(G) \) its vertex set and edge set, respectively. The cardinality \( |V(G)| \) of \( V(G) \) is referred to as the order of G. If \( W \) is a nonempty subset of \( V(G) \), then we denote by \( \langle W \rangle \) the subgraph of G induced by \( W \). A regular graph of degree \( m \) which is a spanning subgraph of G is called an \( m \)-factor of G. It is well-known if G has an \( m \)-factor for some odd \( m \), then the order of G is even. If \( n \) is a positive integer, then by the \( n \)-th power \( G^n \) of G we mean the graph \( G' \) with the properties that \( V(G') = V(G) \) and

\[
E(G') = \{ uv; u, v \in V(G) \text{ such that } 1 \leq d(u, v) \leq n \},
\]

where \( d(w_1, w_2) \) denotes the distance of vertices \( w_1 \) and \( w_2 \) in G.

CHARTRAND, POLIMENI and STEWART [2], and SUMNER [5] proved that if G is a connected graph of even order, then \( G^2 \) has a 1-factor.

The second power of none of the connected graphs in Fig. 1 has a 2-factor. But if G is a connected graph of an order \( p \geq 3 \), then \( G^3 \) has a 2-factor; this follows from a theorem due to SEKANINA [4], which asserts that the third power of any connected graph is hamiltonian connected.

The third power of none of the connected graphs of even order which are given in Fig. 2 has a 3-factor. But for the fourth power the situation is different:

**Theorem.** Let G be a connected graph of an even order \( p \geq 4 \). Then \( G^4 \) has a 3-factor, each component of which is either \( K_4 \) or \( K_2 \times K_3 \).

![Fig. 1.](image_url)
Note that $K_n$ denotes the complete graph of order $n$, and $K_2 \times K_3$ denotes the product of $K_2$ and $K_3$ (see Fig. 3).

Before proving the theorem we establish one lemma. Let $T$ be a nontrivial tree. Consider adjacent vertices $u$ and $v$. Obviously, $T - uv$ has exactly two components, say $T_1$ and $T_2$. Without loss of generality we assume that $u \in V(T_1)$ and $v \in V(T_2)$. Denote $V(T, u, v) = V(T_1)$ and $V(T, v, u) = V(T_2)$.

**Lemma.** Let $T$ be a tree of an order $p \geq 5$. Then there exist adjacent vertices $u$ and $v$ such that

(i) $|V(T, u, v)| \geq 4$ and

(ii) $|V(T, w, u)| \leq 3$ for every vertex $w \neq v$ such that $uw \in E(T)$.

**Proof** of the lemma. Assume that to every pair of adjacent vertices $u$ and $v$ such that $|V(T, u, v)| \geq 4$, there exists a vertex $w \neq v$ such that $uw \in E(T)$ and $|V(T, w, u)| \geq 4$. Since $p \geq 5$, it is possible to find an infinite sequence of vertices $v_0, v_1, v_2, \ldots$ in $T$ such that

(a) $v_0$ has degree one;

(b) $v_0v_1, v_1v_2, v_2v_3, \ldots \in E(T)$;

(c) $v_2 \neq v_0, v_3 \neq v_1, v_4 \neq v_2, \ldots$; and

(d) $|V(T, v_1, v_0)| \geq 4, |V(T, v_2, v_1)| \geq 4, |V(T, v_3, v_2)| \geq 4, \ldots$

Since $T$ is a tree, (b) and (c) imply that the vertices $v_0, v_1, v_2, \ldots$ are mutually different, which is a contradiction. Hence the lemma follows.
Proof of the theorem. Since $G$ is connected, it contains a spanning tree, say $T$.

First, let $p = 4$, $6$, or $8$. If $p = 4$, then $G^4 = T^4 = K_4$.

Let $p = 6$. Then $T$ is isomorphic to one of the six trees of order six (see the list in [3], p. 233). It is easy to see that $T^4$ and therefore $G^4$ contains a 3-factor isomorphic to $K_2 \times K_3$.

Let $p = 8$. By Lemma there exist adjacent vertices $u$ and $v$ of $T$ such that (i) and (ii) hold. If $|V(T, u, v)| = 4$, then $T^4$ (and therefore $G^4$) contains a 3-factor which consists of two disjoint copies of $K_4$. Let $|V(T, u, v)| \geq 5$. Since $p = 8$, we have $|V(T, w, u)| \leq 3$ for every vertex $w$ adjacent to $u$, $w \neq v$. Then there exists a set $R$ of two, three, or four vertices adjacent to $u$ such that

$$\langle \bigcup_{r \in R} V(T, r, u) \rangle_T$$

is isomorphic to one of the graphs $F_1 - F_4$ in Fig. 4. Denote

$$V_R = \bigcup_{r \in R} V(T, r, u).$$

It is clear that $\langle V_R \rangle_{T^4} = K_4$. Since $T - V_R$ is a tree of order four, we conclude that $G^4$ has a 3-factor which consists of two disjoint copies of $K_4$.

Next, let $p \geq 10$. Assume that for every connected graph $G'$ of order $p - 6$ or $p - 4$ we have proved that $(G')^4$ has a 3-factor, each component of which is either $K_4$ or $K_2 \times K_3$. By Lemma there exist adjacent vertices $u$ and $v$ of $T$ such that (i) and (ii) hold. Let $|V(T, u, v)| = 4$ or $6$; then $\langle V(T, u, v) \rangle_{T^4}$ contains a 3-factor isomorphic to either $K_4$ or $K_2 \times K_3$; since $G - V(T, u, v)$ is connected, by the induction assumption $(G - V(T, u, v))^4$ has a 3-factor, each component of which is either $K_4$ or $K_2 \times K_3$; hence $G^4$ has a 3-factor with the required property. Now, let either $|V(T, u, v)| = 5$ or $|V(T, u, v)| \geq 7$. Then there exists a set $S$ of two, three, or four vertices adjacent to $u$ such that

$$\langle \bigcup_{s \in S} V(T, s, u) \rangle_T$$

is isomorphic to one of the graphs $F_1 - F_5$ in Fig. 4. Denote

$$V_S = \bigcup_{s \in S} V(T, s, u).$$

Since $T - V_S$ is a tree, we conclude that $G - V_S$ is a connected graph. According to the induction assumption $(G - V_S)^4$ has a 3-factor, each component of which is
either $K_4$ or $K_2 \times K_3$. Obviously, $|V_S| = 4$ or 6. If $|V_S| = 4$, then $\langle V_S \rangle_{T^4} = K_4$. If $|V_S| = 6$, then it is not difficult to see that $\langle V_S \rangle_{T^4}$ contains a 3-factor which is isomorphic to $K_2 \times K_3$. This implies that $G^4$ has a 3-factor with the required property, which completes the proof.

**Corollary.** Let $G$ be a connected graph of an even order $\geq 4$. Then $G^4$ contains at least three edge-disjoint 1-factors.

**References**


*Author's address*: 116 38 Praha 1, nám. Krasnoarmějců 2 (Filosofická fakulta University Karlovy).