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CONSTRUCTING THE MINIMAL DIFFERENTIAL RELATION WITH PRESCRIBED SOLUTIONS

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Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space, \( \mathcal{K}_n \) the system of its nonempty compact convex subsets, \( \mathcal{K}_n^0 = \mathcal{K}_n \cup \{0\} \).

Let us denote by \( B(x, r), \overline{B}(x, r) \) respectively the open and the closed ball in \( \mathbb{R}^n \) with a centre \( x \) and a radius \( r \).

Given \( M \subset \mathbb{R}^n \), then \( \Omega(M, \varepsilon) \) is the \( \varepsilon \)-neighbourhood of the set \( M, \overline{\Omega}(M, \varepsilon) \) the closure of the neighbourhood. The symbol \( \text{conv} \ M \) stands for the closed convex hull of a set \( M \subset \mathbb{R}^n \), \( m(A) \) is the (one-dimensional) Lebesgue measure of a set \( A \subset \mathbb{R} \).

If \( J \) is an interval, \( M \subset \mathbb{R}^n \), then the upper semicontinuity of a mapping \( F : J \times \times M \to \mathcal{K} \) or \( F : J \times M \to \mathcal{K}_0 \) is defined in the usual way.

Our aim is to prove the following theorem.

**Theorem.** Let \( \alpha < \beta \). Let \( \Xi \) denote a set of functions \( x : J \to \mathbb{R}^n \) with the following properties:

(i) for each \( x \in \Xi \), \( J_x \) is a closed subinterval of \( J = [\alpha, \beta] \);

(ii) \( x \) is absolutely continuous;

(iii) there exists a function \( \xi : [\alpha, \beta] \to \mathbb{R}^+ = [0, +\infty) \) with \( \int_\alpha^\beta \xi(t) \, dt \leq 1 \) such that \( |\dot{x}(t)| \leq \xi(t) \) holds for almost all \( t \in J_x \);

(iv) to each \( x \in \Xi \) there is \( \tau_x \in J_x \) such that \( |x(\tau_x)| \leq 1 \).

Then there exists a mapping \( Q : H \to \mathcal{K}_0 \), where \( H = [\alpha, \beta] \times \overline{B}(0, 2) \), such that \( Q(t, \cdot) \) is upper semicontinuous for almost all \( t \in [\alpha, \beta] \), each \( x \in \Xi \) is a solution of the relation

\[
\dot{x} = Q(t, x)
\]

and \( Q \) is minimal in the following sense: if \( S : H \to \mathcal{K}_n \), \( S(t, \cdot) \) is upper semicontinuous for almost all \( t \in [\alpha, \beta] \) and each \( x \in \Xi \) is a solution (on \( J_x \)) of the relation

\[
\dot{x} \in S(t, x),
\]

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then
\[ Q(t, x) \subset S(t, x) \]
for almost all \( t \in [\alpha, \beta] \) and all \( x \in B(0, 2) \).

Remarks. 1. Let us notice that the minimality property of \( Q \) guarantees its uniqueness.

2. In addition to the upper semicontinuity of \( Q \), it will be clear from the proof that \( Q(t, x) \subset B(0, \xi(t)) \) (cf. condition (iii) of Theorem). Hence \( Q \) satisfies assumptions for the existence of solutions of (1).

3. According to [1, Definition 1.4], a mapping \( F : H \to \mathcal{X}^0_n \) belongs to the class \( \mathcal{S}^n \) if it satisfies the condition: to every \( \varepsilon > 0 \) there is a measurable set \( A_\varepsilon \subset \mathbb{R} \) such that \( m(\mathbb{R} \setminus A_\varepsilon) < \varepsilon \) and the function \( F|_{H \cap (A_\varepsilon \times \mathbb{R}^n)} \) is upper semicontinuous; mappings from \( \mathcal{S}^n \) may be called Scorza-Dragonian mappings as Scorza-Dragoni introduced the corresponding class of functions \( f : H \to \mathbb{R}^n \).

The main result [1, Theorem 1.5] applied to the mapping \( Q : H \to \mathcal{X}^0_n \) with the properties specified in Theorem yields that there exists a Scorza-Dragonian mapping \( Q_0 : H \to \mathcal{X}^0_n \) which fulfils \( Q_0(t, x) \subset Q(t, x) \) for almost all \( t \in [\alpha, \beta] \) and all \( x \in \mathcal{B}(0, 2) \), and each \( u \in \mathcal{E} \) is a solution of the relation
\[ \dot{x} = Q_0(t, x) . \]
Hence necessarily \( Q = Q_0 \), i.e. \( Q \) is Scorza-Dragonian.

Proof of Theorem. If \( x, y \) are two functions satisfying conditions (i)—(iv), let us introduce the distance \( \varrho(x, y) \) in the following way:

Denote by \( J_x = [a_x, b_x] \), \( J_y = [a_y, b_y] \) the definition intervals of \( x \), \( y \), respectively, and set
\[ \bar{x}(t) = \begin{cases} x(t) & \text{for } t \in J_x, \\ x(a_x) & \text{for } \alpha \leq t < a_x, \\ x(b_x) & \text{for } b_x < t \leq \beta ; \end{cases} \]
then \( \bar{x} : J \to \mathbb{R}^n \). Introducing \( \bar{y} : J \to \mathbb{R}^n \) analogously, we define
\[ \varrho(x, y) = \max_{t \in J} |\bar{x}(t) - \bar{y}(t)| + |a_x - a_y| + |b_x - b_y| . \]
It is easily verified that this formula defines a metric on the set of functions satisfying (i)—(iv). We shall show that the set \( \mathcal{E} \) has an at most countable dense (with respect to \( \varrho \)) subset. Indeed, set
\[ \Gamma = \{ x : J \to \mathbb{R}^n \mid x \text{ satisfies (ii), (iii), (iv)} \} . \]
The set \( \mathcal{E} \) with the above defined metric \( \varrho \) is naturally imbedded into the Cartesian product \( \Gamma \times J \times J \). As \( \Gamma \) is separable in virtue of (ii)—(iv), we conclude that \( \mathcal{E} \) is separable as well.
Consequently, there is an at most countable dense subset of \( \mathcal{S} \), say 
\[ V = \{v_1, v_2, \ldots \} \subset \mathcal{S}. \]
Let us denote 
\[ A_i = \{ t \in J \mid \hat{v}_i(t) \text{ does not exist} \}, \quad i = 1, 2, \ldots, \]
\[ A = J - \bigcap_{i=1}^{\infty} A_i. \]
Then \( m(A) = \beta - \alpha. \)

Let us define functions \( Q_i : [\alpha, \beta] \times B(0, 2) \to \mathcal{S}^0, i = 1, 2, \ldots \) by
\[
Q_i(t, x) = \begin{cases} 
\{0\} & \text{for } t \in [\alpha, \beta] - A \\
\overline{\text{conv}} \{ \hat{v}_p(t) \mid v_p(t) \in B(x, i^{-1}) \} & \text{for } t \in A
\end{cases}
\]
and put
\[ Q(t, x) = \bigcap_{i=1}^{\infty} Q_i(t, x). \]

We shall prove that the mapping \( Q \) has the properties from Theorem. First, let us introduce an auxiliary result.

**Lemma.** Let \( x_j : [\alpha, \beta] \to \mathbb{R}^n \) satisfy the assumptions (ii), (iii) of Theorem (with \( x \) replaced by \( x_j \)). Let there exist \( x : [\alpha, \beta] \to \mathbb{R}^n, \)
\[ x(t) = \lim_{j \to \infty} x_j(t) \]
for all \( t \in [\alpha, \beta]. \)
Then
\[ \dot{x}(t) \in \bigcap_{j=1}^{\infty} \overline{\text{conv}} \{ \dot{x}_j(t), \dot{x}_{j+1}(t), \ldots \} \]
for almost all \( t \in [\alpha, \beta]. \)

For this lemma, see [2, p. 395, Theorem D 18.3.10] or [3, Lemma 2].

Now we shall prove that each \( u \in \mathcal{E} \) satisfies the relation
\[ \dot{u}(t) \in Q(t, u(t)) \]
for almost all \( t \in J_u. \)
Indeed, since \( V \) is a set dense in \( \mathcal{S} \), there exists a sequence \( w_j = v_{kj} \in V, j = 1, 2, \ldots, \)
such that
\[ u(t) = \lim_{j \to \infty} w_j(t). \]
According to Lemma there is a set \( A \subset [\alpha, \beta], m(A) = \beta - \alpha, \) such that
\[ \dot{u}(t) \in \bigcap_{j=1}^{\infty} \overline{\text{conv}} \{ \dot{w}_j(t), \dot{w}_{j+1}(t), \ldots \} \]
for all \( t \in A \cap J_u. \)
Given \( t \in A \cap A, \) there exists for every positive integer \( i \) a positive integer \( j \) such that
\[ \overline{\text{conv}} \{ \dot{w}_j(t), \dot{w}_{j+1}(t), \ldots \} \subset Q_i(t, u(t)). \]
(To this aim it is sufficient to choose \( j \) large enough to satisfy \( |w_q(t) - u(t)| \leq i^{-1} \) for all \( q \geq j \).)

Hence
\[
\hat{u}(t) \in Q_i(t, u(t)), \quad i = 1, 2, \ldots
\]
for almost all \( t \) which implies (3) immediately.

Further, we shall prove that the mapping \( Q(t, \cdot) \) is upper semicontinuous for almost all \( t \in [a, \beta] \).

Let us first mention an elementary assertion which is an immediate consequence of the compactness of the sets \( Q_i(t, x), i = 1, 2, \ldots \). For every \( \epsilon > 0 \) there is a positive integer \( i(\epsilon) \) such that
\[
Q_i(t, x) \subseteq Q(Q(t, x), \epsilon)
\]
for all \( i \geq i(\epsilon) \). Indeed, if this were not the case and if \( Q(t, x) \neq 0 \) then we could choose \( \eta > 0 \) and a sequence \( z_i \in Q_i(t, x), |z_i - y| \geq \eta > 0 \) for \( y \in Q(t, x) \). However, passing to a convergent subsequence if necessary we obtain \( z_0 \in Q(t, x) \) for \( z_0 = \lim z_i \), a contradiction. On the other hand, if \( Q(t, x) = 0 \) then \( Q_i(t, x) = 0 \) for \( i \) sufficiently large and (5) is obvious.

Now let \( (t, x_0) \in H \) and \( \epsilon > 0 \). Find \( i(\epsilon) \) so that (5) holds for \( i \geq i(\epsilon) \) and suppose \( |x - x_0| < (2i(\epsilon))^{-1} \), \( z \in Q(t, x) \). Then also \( z \in Q_{2i(\epsilon)}(t, x) \), i.e. for every \( \eta > 0 \) there exists a convex combination
\[
\sum_{j=1}^{p} \beta_j \hat{v}_j(t), \quad \sum_{j=1}^{p} \beta_j = 1, \quad \beta_j > 0
\]
with \( \hat{v}_j \in V \) so that
\[
|z - \sum_{j=1}^{p} \beta_j \hat{v}_j(t)| < \eta
\]
and simultaneously
\[
|x - \sum_{j=1}^{p} \beta_j v_j(t)| \leq \frac{1}{2i(\epsilon)},
\]
hence
\[
|x_0 - \sum_{j=1}^{p} \beta_j v_j(t)| < \frac{1}{i(\epsilon)}.
\]
This means \( z \in Q_{i(\epsilon)}(t, x_0) \). Now we conclude from (5) that
\[
Q(t, x) \subseteq Q_{2i(\epsilon)}(t, x) \subseteq Q_{i(\epsilon)}(t, x_0) \subseteq \Omega(Q(t, x), \epsilon)
\]
provided \( |x - x_0| < \delta = (2i(\epsilon))^{-1} \) which proves the upper semicontinuity of the map \( Q \).

It remains to prove that \( Q \) is minimal in the sense mentioned in the theorem. Let us suppose that \( S \) has the properties from the theorem, i.e. \( S : H \to \mathcal{C}_a^0 \), \( S(t, \cdot) \) is upper semicontinuous for almost all \( t \in [a, \beta] \) and each \( u \in \mathcal{E} \) is a solution of the relation
\[
\dot{x} = S(t, x).
\]

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Let $\varepsilon > 0$, $t \in [\alpha, \beta]$. Then there exists a positive integer $i$ with the following property: if $y \in B(x, i^{-1})$ then

$$S(t, y) \subseteq \Omega(S(t, x), \varepsilon).$$

On the other hand, as the set $V$ is at most countable and all $v_j \in V$ are solutions of (6), there exists a set $D \subseteq [\alpha, \beta]$ with $m(D) = \beta - \alpha$ such that

$$\dot{v}_j(t) \in S(t, v_j(t)) \quad \text{for} \quad t \in D \cap J_{e_j}, \quad j = 1, 2, \ldots.$$

Let $x \in B(0, 1)$, $t \in D \cap A$. Then we have in virtue of the definition of $Q_i$ (see (2))

$$Q_i(t, x) = \overline{\operatorname{conv}} \{v_p(t) \mid v_p(t) \in B(x, i^{-1})\} \subseteq \overline{\operatorname{conv}} \bigcup_p S(t, v_p(t))$$

where the union is taken over all $p$ such that

$$v_p(t) \in B(x, i^{-1}).$$

Consequently, (7) and (9) together imply

$$Q(t, x) = \bigcap_{i=1}^{\infty} Q_i(t, x) \subseteq \Omega(S(t, x), \varepsilon).$$

The number $\varepsilon > 0$ has been arbitrary, hence the last inclusion holds for all $\varepsilon > 0$. This implies immediately $Q(t, x) \subseteq S(t, x)$ for all $t \in D \cap A$, i.e. for almost all $t \in [\alpha, \beta]$ which completes the proof of the theorem.

**References**


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