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INTERSECTION NUMBER OF A GRAPH

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In [1] the intersection numbers $\omega(G)$ and $\omega_0(G)$ of an undirected graph G were defined. Here we shall present some assertions on these numbers. We shall consider finite undirected graphs without loops and multiple edges.

Let G be an undirected graph with a vertex set $V(G)$. Then the intersection number $\omega_0(G)$ (or $\omega(G)$) of G is the minimal cardinality of the set S with the property that there exists a mapping (or an injective mapping, respectively) $\varphi : V(G) \to \exp S - \{\emptyset\}$ such that two vertices v_1, v_2 of G are adjacent if and only if $\varphi(v_1) \cap \varphi(v_2) \neq \emptyset$, (By the symbol $\exp S$ the set of all subsets of S is denoted.)

It is well-known that for every graph G at least one such set S exists. Indeed, for each vertex v of G let $E(v)$ be the set of all edges of G which are incident with v. Then the union of the set of all sets $E(v)$ for non-isolated vertices of G and the set of all isolated vertices of G has the required property; we can put $\varphi(v) = E(v)$ for each non-isolated vertex v and $\varphi(v) = \{v\}$ for each isolated vertex v.

Theorem 1. *The intersection number $\omega_0(G)$ of a finite undirected graph G is equal to the minimal number of cliques of G which cover all vertices and all edges of G.*

Proof. Let $\mathscr{C}$ be the family of cliques of G which cover all vertices and all edges of G, let the cardinality of $\mathscr{C}$ be minimal among all such families. For each vertex v of G let $\varphi(v)$ be the set of all cliques from $\mathscr{C}$ which contain v. If two vertices v_1 and v_2 of G are adjacent, the edge v_1v_2 must be contained in a clique from $\mathscr{C}$. This clique contains both v_1 and v_2 and thus it belongs to $\varphi(v_1) \cap \varphi(v_2)$ and $\varphi(v_1) \cap \varphi(v_2) \neq \emptyset$. If v_1 and v_2 are not adjacent, then there exists no clique in G containing both of them. Thus $\varphi(v_1) \cap \varphi(v_2) = \emptyset$. Therefore we can put $S = \mathscr{C}$ and S has the required properties; we have proved $\omega_0(G) \leqq |\mathscr{C}|$. Now suppose $\omega_0(G) < |\mathscr{C}|$. Then there exists a set S such that $\omega_0(G) = |S| < |\mathscr{C}|$ and S has the property from the definition of $\omega_0(G)$. For each $a \in S$ let $V(a)$ be the set of all vertices x of G such that $a \in \varphi(x)$. Let $w_1 \in V(a), w_2 \in V(a), w_1 \neq w_2$ for some $a \in S$; then we have $a \in \varphi(w_1) \cap \varphi(w_2) \neq \emptyset$ and the vertices w_1, w_2 are adjacent in G. As w_1, w_2 were chosen arbitrarily, we have proved that each $V(a)$ induces a clique in G. For each vertex x of G the set $\varphi(x)$ is non-empty, therefore there exists $a \in \varphi(x)$ and $x \in V(a)$. For each edge $e = x_1x_2$ of G there exists a non-empty set $\varphi(x_1) \cap \varphi(x_2)$ and thus e is contained in the clique induced by the set $V(b)$ for each $b \in \varphi(x_1) \cap \varphi(x_2)$. We have proved that the cliques induced by the sets $V(a)$ form a family of cliques which cover all vertices and all edges of G. The cardinality of this family is equal to $|S|$ and is less than $|\mathscr{C}|$, which is a contradiction with the minimality of $\mathscr{C}$. Therefore $\omega_0(G) = |\mathscr{C}|$.

If G does not contain triangles, then each of its cliques has one or two vertices and the minimal family of cliques which cover all vertices and all edges of G consists of all cliques formed by an edge of G with its end vertices and of all cliques of G formed by an isolated vertex. We have a corollary:

Corollary. *Let G be a finite undirected graph without triangles. Then $\omega_0(G)$ is equal to the sum of the number of edges of G and of the number of isolated vertices of G.*

Now again consider the family $\mathscr{C}$ of cliques and the mapping φ from the proof of Theorem 1. Let $\varepsilon(\mathscr{C})$ be the equivalence on the vertex set $V(G)$ of G defined so that $(x, y) \in \varepsilon(\mathscr{C})$ if and only if $\varphi(x) = \varphi(y)$. Let $\mathscr{D}(\mathscr{C})$ be the family of all equivalences δ such that $\varepsilon(\mathscr{C}) \subseteq \delta$ and each class of δ induces a clique of G. Consider $\delta \in \mathscr{D}(\mathscr{C})$; let $\mathscr{K}(\delta)$ be the family of all equivalence classes of δ. Let K be a class of δ, let $k(K)$ be the maximal cardinality of a class of $\varepsilon(\mathscr{C})$ which is a subset of K. Let $h(\delta) = \sum_{K \in \mathscr{K}(\delta)}]\log_2 k(K)[$; here $]a[$ denotes the least integer which is greater than or equal to a. Let $h(\mathscr{C}) = \min_{\delta \in \mathscr{D}(\mathscr{C})} h(\delta)$. Let $h(G)$ be the minimum of $h(\mathscr{C})$ over all clique families $\mathscr{C}$ satisfying the conditions from the proof of Theorem 1.

Theorem 2. *Let G be a finite undirected graph. Then $\omega(G) \leqq \omega_0(G) + h(G)$.*

Proof. Let $\mathscr{C}$ be the family of cliques with the required properties, such that $h(\mathscr{C}) = h(G)$. Consider the mapping φ with respect to $\mathscr{C}$ as in the proof of Theorem 1. Take $\delta \in \mathscr{D}(\mathscr{C})$ such that $h(\delta) = h(\mathscr{C})$. To each class K of δ assign a set $S(K)$ of the cardinality $]\log_2 k(K)[$ so that $S(K_1) \cap S(K_2) = \emptyset$ for $K_1 \neq K_2$ and $S(K) \cap \mathscr{C} = \emptyset$ for each $K \in \mathscr{K}(\delta)$. Let $S(\delta) = \bigcup_{K \in \mathscr{K}(\delta)} S(K)$, let $S = \mathscr{C} \cup S(\delta)$. Let L be a class of $\varepsilon(\mathscr{C})$ contained in a class K of δ. Let θ_L be an arbitrary injection of L into $\exp S(K)$; such an injection exists, because $|\exp S(K)| \geqq k(K) \geqq |L|$. For each $v \in V(G)$ let $\psi(v) = \varphi(v) \cup \theta_L(v)$, where L is the class of $\varepsilon(\mathscr{C})$ which contains v. Let v_1, v_2 be two vertices of G, $v_1 \neq v_2$. If $\varphi(v_1) \neq \varphi(v_2)$, we have $\psi(v_1) \neq \psi(v_2)$, because $\varphi(v) = \psi(v) \cap \mathscr{C}$ for each v. If $\varphi(v_1) = \varphi(v_2)$, then v_1 and v_2 belong to the same L and $\theta_L(v_1) \neq \theta_L(v_2)$, because θ_L is an injection. As $\theta_L(v_1) = \psi(v_1) \cap S(\delta)$, $\theta_L(v_2) = \psi(v_2) \cap S(\delta)$, we must have $\psi(v_1) \neq \psi(v_2)$ again. We have proved that ψ is an injection. If two vertices v_1, v_2 of G are adjacent, then $\psi(v_1) \cap \psi(v_2) \neq \emptyset$, because $\emptyset \neq \varphi(v_1) \cap \varphi(v_2) \subseteq \psi(v_1) \cap \psi(v_2)$. If v_1, v_2 are not adjacent, then v_1, v_2 belong to distinct classes K_1, K_2 of δ, because each class of δ induces a clique. If L_1, L_2 are the classes of $\varepsilon(\mathscr{C})$ containing v_1 and v_2, respectively, then $L_1 \subseteq K_1$, $L_2 \subseteq K_2$, $\theta_{L_1}(v_1) \subseteq S(K_1)$, $\theta_{L_2}(v_2) \subseteq S(K_2)$. As $S(K_1) \cap S(K_2) = \emptyset$, also $\theta_{L_1}(v_1) \cap \theta_{L_2}(v_2) = \emptyset$. Further, $\varphi(v_1) \cap \cap \varphi(v_2) = \emptyset$ and obviously also $\varphi(v_1) \cap \theta_{L_2}(v_2) = \theta_{L_1}(v_1) \cap \varphi(v_2) = \emptyset$, because $C \cap S(\delta) = \emptyset$. We have obtained the required mapping.

Conjecture. *For every finite undirected graph G we have $\omega(G) = \omega_0(G) + h(G)$.*

Theorem 3. *Let G be a finite undirected graph without triangles and without a connected component containing only one edge. Then $\omega(G) = \omega_0(G)$.*

Proof. As was mentioned above, the minimal family of cliques which cover all vertices and all edges of G consists of all two-vertex cliques formed by an edge of G with its end vertices and of all one-vertex cliques formed by an isolated vertex. If v_1, v_2 are distinct vertices of G, then either they are both isolated, or one of them is incident with an edge which is not incident with the other. Therefore the equivalence $\varepsilon(\mathscr{C})$ is the identity relation on $V(G)$ and $h(G) = 0$. This implies the assertion.

Reference

[1] *F. Harary - E. Palmer:* Graphical Enumeration. New York—London 1973.

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