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JOINS OF CONGRUENCES IN Ω-GROUPS

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The symmetric and transitive relations in a set G (also called *partitions in* G) form a complete lattice with respect to the set theoretic inclusion; it is denoted by $P(G)$. If G is a universal algebra, the *congruences in* G, i.e. stable partitions in G, also form a complete lattice with respect to the same ordering. This lattice is denoted by $\mathscr{K}(G)$ and is a closed $\wedge$-subsemilattice of $P(G)$. The joins $\vee_P$ and $\vee_{\mathscr{K}}$ in these lattices do not coincide in general – in contrast to the joins in the lattice $\pi(G)$ of all *partitions on* G (i.e. reflexive partitions in G) and in the lattice $\mathscr{C}(G)$ of *congruences on* G (stable partitions on G), it is namely $\vee_P = \vee_\pi = \vee_{\mathscr{C}}$. Naturally, the P-join of any two partitions B and C in an algebra G does not depend on any algebraic structure defined on the set G, so even the $\mathscr{C}$-join of congruences on the algebra G does not depend on it. In more detail, what is understood by the notion of independence of the join of congruences on an algebraic structure defined on G: Let B and C be congruences on an algebra G with a system of operations F_1 and $\mathscr{C}_1$ the lattice of all congruences on (G, F_1). If F_2 is another system of operations on the set G, $\mathscr{C}_2$ the lattice of all congruences on the algebra (G, F_2) and $B, C \in \mathscr{C}_2$, then $B \vee_{\mathscr{C}_1} C = B \vee_{\mathscr{C}_2} C$.

We shall be interested in a less restricted problem, namely for $F_1 = \emptyset$, i.e. in searching those pairs B and C of congruences in an Ω-group G, $\mathscr{K}$-join of which does not depend on the given algebraic structure defined on the set G. Thus we shall investigate properties characterizing pairs B and C of congruences in an Ω-group G with the property $B \vee_P C = B \vee_{\mathscr{K}} C$, and some related problems. We leave the problem of the stronger independence of joins mentioned above open.

We review some of the notation and theory that is needed. A more detailed information may be found in [1–4], especially as to congruences in algebras, see [1] I.

Given a binary relation A in a set G and $x \in G$ we define $A(x) = \{y \in G : yAx\}$ and $\bigcup A = \bigcup\{A(x) : x \in G\}$ ([1] 3.5). If A is a symmetric and transitive relation in G (i.e. a *partition in* G) and $A(x) \neq \emptyset$, then the set $A(x)$ is said to be the *block* of the partition A and the set $\bigcup A$ the *domain* of the partition A.

Let G be an algebra. Then $\mathscr{K}(G)$ is a complete lattice with respect to the ordering by inclusion. For $\{A_\alpha\} \subseteq \mathscr{K}(G)$ we have $\bigwedge_{\alpha}{}_{\mathscr{K}} A_\alpha = \bigcap_\alpha A_\alpha$. If G is an Ω-group then the set of all nonempty congruences in G is a closed sublattice of the lattice $\mathscr{K}(G)$, [1] 1.1. Let A be a (nonempty) congruence in G. Then $\bigcup A$ is an Ω-subgroup of G, $A(0)$ an ideal of $\bigcup A$ and $A = \bigcup A/A(0)$, [1] 1.4. If $\{A_\alpha\} \subseteq \mathscr{K}(G)$ then $\bigcup(\bigvee_{\mathscr{K}} A_\alpha) = \langle \bigcup(\bigcup_\alpha A_\alpha)\rangle$ and $(\bigvee_{\mathscr{K}} A_\alpha)(0) = \langle\langle \bigcup_\alpha A_\alpha(0)\rangle\rangle_{\mathfrak{A}}$, where $\mathfrak{A} = \langle \bigcup(\bigcup_\alpha A_\alpha)\rangle$ is the Ω-subgroup generated by the set $\bigcup(\bigcup_\alpha A_\alpha)$ and $\langle\langle \bigcup_\alpha A_\alpha(0)\rangle\rangle_{\mathfrak{A}}$ is the ideal generated in $\mathfrak{A}$ by the set $\bigcup_\alpha A_\alpha(0)$, [1] 1.6.

The results of the present paper are based on Lemma 1.6, in which a description of blocks of the partition $B \vee_P C$ is given, where B and C are congruences in G and G is an Ω-group. Further, criteria are given for the validity of the following identities:

$(B \vee_P C) \sqsupset (\bigcup B \cap \bigcup C) = (B \vee_{\mathscr{K}} C) \sqsupset (\bigcup B \cap \bigcup C)$ (Theorem 2.3),
$B \vee_P C = B \vee_{\mathscr{K}} C$ (Theorems 2.5 and 2.7),
$(B \vee_{\mathscr{K}} C) \sqsupset (\bigcup B \cup \bigcup C) = B \vee_P C$ (Theorem 2.9),
$(B \vee_{\mathscr{K}} C) \sqcap (\bigcup B \cup \bigcup C) = B \vee_P C$ (Theorems 2.11 and 2.12),

where the partition $A \sqsupset \mathfrak{A}$ (or $\mathfrak{A} \sqsubset A$), called the *closure* of the subset $\mathfrak{A}(\subseteq G)$ in the partition $A(A \in P(G))$, is the set of all blocks of A that are incident with $\mathfrak{A}$ and $A \sqcap \mathfrak{A}$ $(= \mathfrak{A} \sqcap A) = \{A^1 \cap \mathfrak{A} : A^1 \in A,\ A^1 \cap \mathfrak{A} \neq \emptyset\}$ (called the *intersection* of the partition A and the subset $\mathfrak{A}$) – see [4] 2.3.

In what follows G will denote an Ω-group and B and C (nonempty) congruences in G, unless otherwise indicated.

1.1 Lemma. *If $x \in \bigcup B \cap \bigcup C$ then $BC(x) = x + BC(0) = BC(0) + x$.*

Proof. For $x \in \bigcup B \cap \bigcup C$ we have

$$y \in BC(0) + x \Leftrightarrow y - x \in BC(0) \Leftrightarrow \exists a \in G\,,\ (y - x) BaC0 \Leftrightarrow$$
$$\Leftrightarrow \exists a \in G\,,\ yB(a + x)\,Cx \Leftrightarrow yBCx \Leftrightarrow y \in BC(x)\,.$$

Similarly $y \in x + BC(0) \Leftrightarrow y \in BC(x)$.

1.2 Lemma. *If $x \in \bigcup B \cap \bigcup C$ then $BCB \ldots (x) = BC(x)$, where the product on the left-hand side contains a finite number ($\geqq 2$) of factors.*

Proof by induction on the number n of factors. It suffices to show $BCB \ldots (x) \subseteq BC(x)$ for $x \in \bigcup B \cap \bigcup C$, because the converse inclusion is evident. In fact, if $x \in \bigcup B \cap \bigcup C$ then $yBCx \Rightarrow yBCxBxCx \ldots x \Rightarrow y(BCB \ldots)\,x$.

The inclusion $\subseteq$ is valid for $n = 2$. First, we shall prove it for $n = 3$. If $x \in \bigcup B \cap \bigcup C$ then

$$y \in BCB(x) \Leftrightarrow \exists a \in G\,,\ yBCaBx \Leftrightarrow (\text{by 1.1})\ \exists a \in G,\ y \in a + BC(0)\,,\ a \in x + B(0) \Leftrightarrow$$
$$\Leftrightarrow y \in x + B(0) + BC(0) =$$
$$= x + B(0) + B(0) + \bigcup B \cap C(0) = x + BC(0)$$

(by [1] 3.5.5). By 1.1, the last expression is equal to $BC(x)$. The inductive hypothesis: Let $n \geqq 4$ and let $x \in \bigcup B \cap \bigcup C$ imply $BCB \ldots (x) \subseteq BC(x)$, whenever the number p of factors on the left-hand side fulfils $2 < p \leqq n - 1$. Now, let $x \in \bigcup B \cap \bigcup C$, $yBCB \ldots x$ and let the product contain $n \geqq 4$ factors. Then there exists $a \in \bigcup B \cap \bigcup C$ such that $yBCa(BC \ldots) x$, thus by 1.1 $y - a \in BC(0)$. By assumption $a \in BCB \ldots \ldots (x) \subseteq BC(x)$, hence $y = (y - a) + a \in BC(0) + BC(x) \subseteq BC(x)$. The last inclusion follows from the implication $tBC0$, $zBCx \Rightarrow (t + z) BCx$. So we have got that $x \in \bigcup B \cap \bigcup C$ satisfies $BCB \ldots (x) \subseteq BC(x)$, which was to be proved.

1.3 Lemma. *If* $x \in \bigcup B \cap \bigcup C$ *then*

$$B \vee_P C(0) = B(0) \cup BC(0) \cup C(0) \cup CB(0) =$$
$$= [B(0) + \bigcup B \cap C(0)] \cup [C(0) + \bigcup C \cap B(0)],$$
$$B \vee_P C(x) = B(x) \cup BC(x) \cup C(x) \cup CB(x) =$$
$$= x + B \vee_P C(0) = B \vee_P C(0) + x .$$

The member in the first square bracket or in the second one is an ideal of the Ω-group $B(0) + \bigcup B \cap \bigcup C$ or $C(0) + \bigcup B \cap \bigcup C$, respectively. The order of summands (in one or both the square brackets) may be changed.

Proof. The first assertion is Corollary 3.5.7 [1]. Proof of the second one follows by a similar agument: Denote $B_n = BCB \ldots$ and $C_n = CBC \ldots$, provided the product on the right-hand sides contains n $(\geqq 1)$ factors. Now, the assertion follows from 1.2 and 1.1 because

$$B \vee_P C(x) = \bigcup_{n=1}^{\infty} B_n(x) \cup \bigcup_{n=1}^{\infty} C_n(x) = B(x) \cup BC(x) \cup C(x) \cup CB(x) =$$
$$= [x + B(0)] \cup [x + BC(0)] \cup [x + C(0)] \cup [x + CB(0)] =$$
$$= x + [B(0) \cup C(0) \cup BC(0) \cup CB(0)] = x + B \vee_P C(0) .$$

Analogously we obtain the identity $B \vee_P C(x) = B \vee_P C(0) + x$.

1.4 A generalization of the first assertion of Lemma 1.3 for an arbitrary number of congruences will be given in the following

Theorem. *Let B_α $(\alpha \in A)$ be congruences in G. Then*

$$\left(\bigvee_{\alpha \in A} {}_P B_\alpha\right)(0) = \bigcup_{n=1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_n} W(\alpha_1, \ldots, \alpha_n),$$

where $W(\alpha_1) = B_{\alpha_1}(0)$, $W(\alpha_1, \ldots, \alpha_n) = W(\alpha_1, \ldots, \alpha_{n-1}) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0)$, $n = 2, 3, \ldots$ *and* $\alpha_1, \ldots, \alpha_n$ *is an n-tuple of elements of* A.

Note. In the definition of $W(\alpha_1, \ldots, \alpha_n)$ it is possible to interchange both the summands (because $B_{\alpha_n}(0)$ is an ideal of $\bigcup B_{\alpha_n}$).

Proof. Denote $V = \bigvee_{P,\, \alpha \in A} B_\alpha$ and let W stands for the expression on the right-hand side of the required identity. Let $y \in V(0)$. Then $0B_{\alpha_1}y_1B_{\alpha_2}y_2 \ldots y_{n-1}B_{\alpha_{n-1}}y_nB_{\alpha_n}y$ for suitable $y_1, \ldots, y_n \in G$ and $\alpha_1, \ldots, \alpha_n \in A$.

Hence

$$y_1 \in B_{\alpha_1}(0) = W(\alpha_1)\,, \quad y_2 \in y_1 + B_{\alpha_2}(0) \subseteq B_{\alpha_1}(0) \cap \bigcup B_{\alpha_2} + B_{\alpha_2}(0) =$$
$$= W(\alpha_1) \cap \bigcup B_{\alpha_2} + B_{\alpha_2}(0) = W(\alpha_1, \alpha_2), \ldots, y_i = y_{i-1} + B_{\alpha_i}(0) \subseteq$$
$$\subseteq W(\alpha_1, \ldots, \alpha_{i-1}) \cap \bigcup B_{\alpha_i} + B_{\alpha_i}(0) = W(\alpha_1, \ldots, \alpha_i), \ldots, y \in y_n + B_{\alpha_n}(0) \subseteq$$
$$\subseteq W(\alpha_1, \ldots, \alpha_{n-1}) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) = W(\alpha_1, \ldots, \alpha_n)\,.$$

Therefore $V(0) \subseteq W$. Now let $\alpha_1, \ldots, \alpha_n$ be an arbitrary n-tuple of elements of A. Then $W(\alpha_1, \ldots, \alpha_n) \subseteq V(0)$. In fact, if $n = 1$ then $W(\alpha_1) = B_{\alpha_1}(0) \subseteq V(0)$. We use induction on n. By the inductive hypothesis $W(\alpha_1, \ldots, \alpha_{n-1}) \subseteq V(0)$ we have

$$W(\alpha_1, \ldots, \alpha_n) = W(\alpha_1, \ldots, \alpha_{n-1}) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) \subseteq$$
$$\subseteq V(0) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) \subseteq V(0)\,.$$

We obtain the last inclusion as follows. For $v \in V(0) \cap \bigcup B_{\alpha_n}$ the block $v + B_{\alpha_n}(0)$ of the partition B_{α_n} meets $V(0)$, so $v + B_{\alpha_n}(0) \subseteq V(0)$ for all $v \in V(0) \cap \bigcup B_{\alpha_n}$ and thus

$$V(0) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) \subseteq V(0)\,, \quad W(\alpha_1, \ldots, \alpha_n) \subseteq V(0) \quad \text{and} \quad W \subseteq V(0)\,.$$

Finally $V(0) \subseteq W \subseteq V(0)$, hence $V(0) = W$.

We can obtain the first assertion of Lemma 1.3 as a special case of Theorem 1.4 in the following way. Denote $B = B_1$ and $C = B_2$. Evidently $W(1) = B_1(0) \subseteq B_2(0) \cap \bigcup B_1 + B_1(0) = W(2, 1)$; analogously $W(2) \subseteq W(1, 2)$. Further $W(1, 2, 1) = [B_1(0) \cap \bigcup B_2 + B_2(0)] \cap \bigcup B_1 + B_1(0) = [B_1(0) \cap \bigcup B_2 + B_2(0) + B_1(0)] \cap \bigcup B_1 = [B_2(0) + B_1(0)] \cap \bigcup B_1 = B_2(0) \cap \bigcup B_1 + B_1(0) = W(2, 1)$.

Similarly $W(2, 1, 2) = W(1, 2)$, $W(1, 1) = W(1)$ and $W(2, 2) = W(2)$. Iterating this procedure we obtain $V(0) = \bigcup_{n=1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_n} W(\alpha_1, \ldots, \alpha_n) = W(1, 2) \cup W(2, 1)$ which is the required assertion.

1.5 In the next theorem, another construction of the set $(\bigvee_{P,\, \alpha \in A} B_\alpha)(0)$ is given.

Theorem. *Let B_α $(\alpha \in A)$ be congruences in G. Then*

$$\big(\bigvee_{P\, \alpha \in A} B_\alpha\big)(0) = \bigcup_{\alpha \in A} \{\bigcup B_\alpha \cap [\![\bigcup_{\beta \in A} B_\beta(0)]\!] + B_\alpha(0)\}\,.$$

Note. It is possible to interchange the summands on the right-hand side (because $B_\alpha(0)$ is an ideal of $\bigcup B_\alpha$). Again, on the right-hand side, $\bigcup\limits_{\beta\in A,\beta\neq\alpha}$ can be put in place of $\bigcup\limits_{\alpha\in A}$. The symbol $[\![\mathfrak{A}]\!]$ denotes the subgroup of G generated by the subset $\mathfrak{A}$ of G.

Proof. Denote $V = \bigvee_{P,\ \alpha\in A} B_\alpha$. We have

$$\bigcup_{\alpha\in A}\{\bigcup B_\alpha \cap [\![\bigcup_{\beta\in A} B_\beta(0)]\!] + B_\alpha(0)\} =$$

$$= \bigcup_{\alpha\in A}\{\bigcup B_\alpha \cap \bigcup_{n=1}^{\infty}\ \bigcup_{\alpha_1,\dots,\alpha_n} (B_{\alpha_n}(0) + \dots + B_{\alpha_1}(0)) + B_\alpha(0)\} =$$

$$= \bigcup_{\alpha\in A}\ \bigcup_{n=1}^{\infty}\ \bigcup_{\alpha_1,\dots,\alpha_n} \{\bigcup B_\alpha \cap [\![B_{\alpha_n}(0), \dots, B_{\alpha_1}(0)]\!] + B_\alpha(0)\},$$

where $\alpha_1, \dots, \alpha_n$ runs through all n-tuples of elements of A. We shall show that

$$\bigcup B_\alpha \cap [\![B_{\alpha_n}(0), \dots, B_{\alpha_1}(0)]\!] + B_\alpha(0) \subseteq V(0),$$

and so the inclusion $\supseteq$ in the assertion of Theorem will be proved.

Thus, let $b_1, \dots, b_k$ be arbitrary elements of the set $B_{\alpha_1}(0) \cup \dots \cup B_{\alpha_k}(0)$ with $b_k + \dots + b_1 \in \bigcup B_\alpha$ and let $b \in B_\alpha(0)$. If $k = 1$ then $b_1 + b \in V(0)$, since the block $b_1 + B_\alpha(0)$ of the partition B_α meets $V(0)$ and $B_\alpha \leqq V$. We use induction on k. Suppose that $b_1, \dots, b_{k+1} \in B_{\alpha_1}(0) \cup \dots \cup B_{\alpha_{k+1}}(0)$, $b_{k+1} + \dots + b_1 \in \bigcup B_\alpha$, $b \in B_\alpha(0)$ and $b_k + \dots + b_1 + b \in V(0)$. Then $b_{k+1} + \dots + b_1 + b \in (B_{\alpha_{k+1}}(0) + b_k + \dots + b_1 + b) \subseteq B_{\alpha_{k+1}}(0) + V(0)$ for some α_{k+1}. Since for an arbitrary $v \in V(0)$ the block $B_{\alpha_{k+1}}(0) + v$ of the partition $B_{\alpha_{k+1}}$ meets $V(0)$ then $B_{\alpha_{k+1}}(0) + v \subseteq V(0)$, whence $B_{\alpha_{k+1}}(0) + V(0) \subseteq V(0)$. Finally $b_{k+1} + \dots + b_1 + b \in V(0)$ which completes the proof by induction.

1.6 In the next lemma the description of blocks of the partition $B \vee_P C$ is given. This is the crucial lemma for our study.

Lemma. *The following implications hold*:

(1) $x \in \bigcup B \cap \bigcup C \Rightarrow B \vee_P C(x) = x + B \vee_P C(0) = B \vee_P C(0) + x$,

(2) $x \in \bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)] \Rightarrow B \vee_P C(x) = x + B(0) = B(0) + x = B(x)$,

(3) $x \in \bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)] \Rightarrow B \vee_P C(x) = x + C(0) = C(0) + x = C(x)$.

The blocks (1) are exactly the blocks of the partition $(B \vee_P C) \sqsupset \bigcup B \cap \bigcup C$, the domain of which is $(B(0) \cup C(0)) + (\bigcup B \cap \bigcup C)$; the blocks (2) and (3) are the remaining blocks of the partition $B \vee_P C$. The blocks (2) cover the set $\bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)]$, and the blocks (3) cover the set $\bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)]$.

Proof. (1) follows from 1.3. Thus the system of sets $\{B \vee_P C(0) + x : x \in \bigcup B \cap \bigcup C\}$ is equal to the set of the blocks of the partition $(B \vee_P C) \sqsupset (\bigcup B \cap \bigcup C)$. The domain $\mathfrak{X}$ of this partition is $\mathfrak{X} = B \vee_P C(0) + (\bigcup B \cap \bigcup C) = \{[B(0) + \bigcup B \cap C(0)] \cup [C(0) + \bigcup C \cap B(0)]\} + (\bigcup B \cap \bigcup C) = (B(0) \cup C(0)) + (\bigcup B \cap \bigcup C)$.

If $x \in \bigcup B \setminus \mathfrak{X}$, then $B \vee_P C(x) = B(x)$ and if $x \in \bigcup C \setminus \mathfrak{X}$, then $B \vee_P C(x) = C(x)$. Finally let us recall that $\bigcup B \setminus \mathfrak{X} = \bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)]$, as $[C(0) + (\bigcup B \cap \bigcup C)] \cap \bigcup B = \bigcup B \cap \bigcup C$. Analogously $\bigcup C \setminus \mathfrak{X} = \bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)]$.

2.1 Definition. $\langle \bigcup B, \bigcup C \rangle$ is the Ω-subgroup generated in G by the set $\bigcup B \cup \bigcup C$ and $\langle\!\langle B(0), C(0) \rangle\!\rangle_{\mathfrak{A}}$ is the ideal generated in $\mathfrak{A} = \langle \bigcup B, \bigcup C \rangle$ by the set $B(0) \cup C(0)$.

2.2 Lemma. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be subgroups of a group G. If $\mathfrak{A} \cup \mathfrak{B} = \mathfrak{C}$, then the sets $\mathfrak{A}$ and $\mathfrak{B}$ are comparable by inclusion.*

Proof. If the sets $\mathfrak{A}$ and $\mathfrak{B}$ are incomparable by inclusion, then there exists elements $x \in \mathfrak{A} \setminus \mathfrak{B}$ and $y \in \mathfrak{B} \setminus \mathfrak{A}$ and it holds $C \ni \mathbf{x} + \mathbf{y} \bar{\in} \mathfrak{A} \cup \mathfrak{B} = \mathfrak{C}$, a contradiction.

2.3 Theorem. *The identity*

$$(1) \qquad (B \vee_P C) \sqsupset (\bigcup B \cap \bigcup C) = (B \vee_{\mathscr{X}} C) \sqsupset (\bigcup B \cap \bigcup C)$$

holds if and only if $B(0) \subseteq \bigcup C$ or $C(0) \subseteq \bigcup B$, and simultaneously $B(0) + C(0)$ is an ideal of $\langle \bigcup B, \bigcup C \rangle$.

Proof. The condition (1) is equivalent to the following one:

$$B \vee_P C(x) = B \vee_{\mathscr{X}} C(x) \quad \text{for each} \quad x \in \bigcup B \cap \bigcup C\,.$$

By 1.3, if $x \in \bigcup B \cap \bigcup C$ then

$$B \vee_P C(x) = x + B \vee_P C(0) =$$

$$= x + \{[B(0) + \bigcup B \cap C(0)] \cup [\bigcup C \cap B(0) + C(0)]\}$$

and further

$$B \vee_{\mathscr{X}} C(x) = x + \langle\!\langle B(0), C(0) \rangle\!\rangle_{\mathfrak{A}}\,, \quad \text{where} \quad \mathfrak{A} = \langle \bigcup B, \bigcup C \rangle \quad ([1]\ 1.6)\,.$$

Then the identity $B \vee_P C(x) = B \vee_{\mathscr{X}} C(x)$ is true for each $x \in \bigcup B \cap \bigcup C$ if and only if the following identity (2) is valid:

$$(2) \qquad [B(0) + \bigcup B \cap C(0)] \cup [\bigcup C \cap B(0) + C(0)] = \langle\!\langle B(0), C(0) \rangle\!\rangle_{\mathfrak{A}}\,.$$

Let (2) hold. The left-hand side of (2) is the union of two Ω-subgroups. Denote them by $\mathfrak{A}$ and $\mathfrak{B}$. The right-hand side is an Ω-subgroup. By 2.2 the sets $\mathfrak{A}$ and $\mathfrak{B}$ are comparable. Thus we have e.g.

$$(3) \qquad B(0) + \bigcup B \cap C(0) \subseteq \bigcup C \cap B(0) + C(0)\,.$$

The right-hand side is a subset of $\bigcup C$, hence $B(0) \subseteq \bigcup C$. Now, (2) has the form

$$(4) \qquad B(0) + C(0) = \langle\!\langle B(0), C(0) \rangle\!\rangle_{\mathfrak{A}}\,.$$

It follows that $B(0) + C(0)$ is an ideal of $\langle \bigcup B, \bigcup C \rangle$.

If we start from the converse inclusion in (3) we obtain $C(0) \subseteq \bigcup B$ and (4).

To prove the converse implication it suffices to verify that (2) is true whenever the conditions of Theorem are fulfilled. The left-hand side of (2) is equal to $B(0) + C(0)$, and by supposition, this is an ideal of $\langle \bigcup B, \bigcup C \rangle$, hence $B(0) + C(0) = \langle\!\langle B(0), C(0) \rangle\!\rangle_{\mathfrak{A}}$. This completes the proof.

2.4 Corollary. *If B and C are congruences on G then $B \vee_P C = B \vee_{\mathscr{K}} C$ $(= B \vee_{\mathscr{C}} C)$.*

2.5 If we investigate conditions which guarantee the validity of the identity $B \vee_P C = B \vee_{\mathscr{K}} C$ for congruences B and C in G, we may restrict ourselves to incomparable congruences, because comparable congruences fulfil it evidently.

. Theorem. *If B and C are incomparable congruences in G then*

$$(5) \qquad B \vee_P C = B \vee_{\mathscr{K}} C$$

if and only if

$$(6) \qquad B(0) + \bigcup C = \bigcup B \quad \text{or} \quad \bigcup B + C(0) = \bigcup C$$

or equivalently if

$$(7) \qquad \bigcup(B \vee_P C) = \bigcup(BC) \quad \text{or} \quad = \bigcup(CB)$$

or equivalently if

$$(8) \qquad \bigcup(B \vee_{\mathscr{K}} C) = \bigcup(BC) \quad \text{or} \quad = \bigcup(CB)\,.$$

Note. Due to the symmetry between B and C in (5) the summands in (6) can be interchanged.

Proof. 5 ⇒ 6. Because $\bigcup B \cup \bigcup C = \langle \bigcup B, \bigcup C \rangle$, 2.2 implies that either $\bigcup B \supseteq \bigcup C$ or $\bigcup B \subseteq \bigcup C$, say $\bigcup B \supseteq \bigcup C$. Then we have $B(0) + \bigcup C \subseteq \bigcup B$. If $\neq$ then there exists $x \in \bigcup B \setminus [B(0) + \bigcup C]$ nad by 1.6, this x satisfies $B \vee_P C(x) = B(x) = x + B(0)$. Since $B \vee_{\mathscr{K}} C(x) = x + \langle\!\langle B(0), C(0) \rangle\!\rangle_{\cup B}$, (5) implies $B(0) = \langle\!\langle B(0), C(0) \rangle\!\rangle_{\cup B}$, thus $B(0) \supseteq C(0)$ and finally $B \geqq C$, a contradiction. Analogously, if $\bigcup C \supseteq \bigcup B$, then $\bigcup B + C(0) = \bigcup C$.

$6 \Rightarrow 5$. Let $B(0) + \bigcup C = \bigcup B$ be true. We shall prove that $B(0) + C(0)$ is an ideal of $\bigcup B = \langle \bigcup B, \bigcap C \rangle$. The proof is based on the elementary procedures to follow. Denote by b or $\bar{b}$ (with indices if necessary) elements of $\bigcup B$ or $B(0)$, respectively. Similarly for $\bigcup C$ and $C(0)$. The set $B(0) + C(0)$ is an Ω-subgroup (since $C(0) \subseteq \subseteq \bigcup B$). We shall show it is normal in $\bigcup B$. Arbitrary elements $b, \bar{b}, \bar{c}$ and suitable elements $\bar{b}', \bar{b}'', \bar{b}''', c, \bar{c}'$ satisfy $b + \bar{b} + \bar{c} - b = \bar{b}' + c + \bar{b} + \bar{c} - c - \bar{b}' = = \bar{b}' + \bar{b}'' + c + \bar{c} - c - \bar{b}' = \bar{b}' + \bar{b}'' + \bar{b}''' + \bar{c}' \in B(0) + C(0)$.

If ω is an n-ary operation in G we shall shortly write $g_i\omega$ instead of $g_1 \dots g_n\omega$. For arbitrary elements $b_i, \bar{c}_i, \bar{b}_i$ and suitable elements $\bar{b}, \bar{b}', \bar{b}'', \bar{b}''', c_i, \bar{c}'$ we have $b_i\omega = (\bar{b}_i + c_i)\,\omega = c_i\omega + \bar{b}$, $(b_i + \bar{c}_i + \bar{b}_i)\,\omega = (b_i + \bar{c}_i)\,\omega + \bar{b}' = (\bar{b}_i + c_i + + \bar{c}_i)\,\omega + \bar{b}' = (c_i + \bar{c}_i)\,\omega + \bar{b}'' + \bar{b}' = c_i\omega + \bar{c}' + \bar{b}'' + \bar{b}'$.

Hence

$$-b_i\omega + (b_i + \bar{c}_i + \bar{b}_i)\,\omega =$$
$$= -\bar{b} - c_i\omega + c_i\omega + \bar{c}' + \bar{b}'' + \bar{b}' = \bar{b}''' + c' \in B(0) + C(0)\,.$$

So we have shown that $B(0) + C(0)$ is an ideal of $\bigcup B$.

By 2.3, $(B \vee_P C) \sqsupset \bigcup C = (B \vee_{\mathscr{K}} C) \sqsupset \bigcup C$ is true. By 1.6, $(B \vee_P C) \sqsupset \bigcup C = = B \vee_P C$ holds and the identity $B(0) + \bigcup C = \bigcup B$ $(= \bigcup(B \vee_{\mathscr{K}} C))$ yields $(B \vee_{\mathscr{K}} C) \sqsupset \bigcup C = B \vee_{\mathscr{K}} C$. This completes the proof of $6 \Rightarrow 5$. The remaining part of the assertion follows from [1] 3.7.5.

2.6 Corollary. ([1] 3.11) *If* $\bigcup B = \bigcup C$ *then* $B \vee_P C = B \vee_{\mathscr{K}} C$.

Proof follows from 2.5 since $B(0) \subseteq \bigcup B = \bigcup C$ implies $B(0) + \bigcup C = \bigcup B$.

The converse implication is true for commuting congruences.

2.7 Corollary. *If B and C commute and $B \parallel C$ then*

$$B \vee_P C = B \vee_{\mathscr{K}} C \quad \textit{if and only if} \quad \bigcup B = \bigcup C\,.$$

Proof. $\Rightarrow$: If B and C commute then [1] 3.9 yields $B(0) \cup C(0) \subseteq \bigcup B \cap \bigcup C$ and by 2.5 the condition (6) is fulfilled. This condition gives $\bigcup B = \bigcup C$.

The converse follows from 2.6.

2.8 Proposition. *Let G be an Ω-group. Then the following conditions are equivalent*:

(a) *The lattice $\mathscr{K}(G)$ is a sublattice of the lattice $P(G)$.*
(b) *$\mathscr{K}(G)$ is a chain.*
(c) *$\mathscr{K}(G)$ has three elements only, G/G, $G/\{0\}$ and $\{0\}/\{0\}$.*
(e) *G has no proper Ω-subgroups.*

Note. If G is a group then the condition (e) reads: *G is a cyclic group of prime order.*

Proof. a $\Rightarrow$ d. Let $\mathfrak{A}$ be a proper Ω-subgroup of G, $B = G/\{0\}$ and C an arbitrary congruence in G with $\bigcup C = \mathfrak{A}$. If $C(0) \neq \{0\}$ then B and C are incomparable, thus $\mathfrak{A} = G$ by 2.5, a contradiction. Hence $C(0) = \{0\}$. In particular, for $C = \mathfrak{A}/\mathfrak{A}$ we have $C(0) = \mathfrak{A} = \{0\}$, a contradiction. Therefore G has no proper Ω-subgroups.

$d \Rightarrow c \Rightarrow b \Rightarrow a$ is evident.

2.9 Theorem. *The identity*

$$(B \vee_{\mathscr{K}} C) \sqsupset (\bigcup B \cup \bigcup C) = B \vee_P C \tag{9}$$

holds if and only if

$$B(0) = C(0) \textit{ is an ideal of } \langle \bigcup B, \bigcup C \rangle \quad \textit{or} \quad B \vee_{\mathscr{K}} C = B \vee_P C\,. \tag{10}$$

Note. The condition (9) reads that the set of all blocks of the partition $B \vee_P C$ is a subset of the set of all blocks of the partition $B \vee_{\mathscr{K}} C$. These blocks of the partition $B \vee_{\mathscr{K}} C$ cover the domain $\bigcup B \cup \bigcup C$ of the partition $B \vee_P C$.

Proof. Denote $\mathfrak{D} = (B \vee_{\mathscr{K}} C)(0)$ and suppose (9). By 1.3, $\mathfrak{D} = (B \vee_{\mathscr{K}} C)(0) = (B \vee_P C)(0) = [B(0) + \bigcup B \cap C(0)] \cup [C(0) + \bigcup C \cap B(0)] \subseteq B(0) + C(0) \subseteq \mathfrak{D}$, thus $\mathfrak{D} = B(0) + C(0) = [B(0) + \bigcup B \cap C(0)] \cup [C(0) + \bigcup C \cap B(0)]$. The left-hand side is a subgroup, the right-hand side is the union of two subgroups. By 2.2 we have e.g.

$$B(0) + \bigcup B \cap C(0) \subseteq C(0) + \bigcup C \cap B(0)\,. \tag{11}$$

The right-hand side is contained in $\bigcup C$, hence $B(0) \subseteq \bigcup C$. Denote $G_0 = \bigcup B \cap \bigcup C$. Then etiher $\bigcup B \setminus (\mathfrak{D} + G_0) \neq \emptyset$, hence $B(0) = (B \vee_{\mathscr{K}} C)(0) \supseteq C(0)$ by 1.6 and (9), hence $B(0) \supseteq C(0)$, or $\bigcup B \subseteq \mathfrak{D} + G_0$, thus $\bigcup B \subseteq C(0) + B(0) + \bigcup B \cap \bigcup C = C(0) + \bigcup B \cap \bigcup C \subseteq \bigcup C$ by (11). Hence $\bigcup B \subseteq \bigcup C$.

Simultaneously either $\bigcup C \setminus (\mathfrak{D} + G_0) \neq \emptyset$, then $C(0) = (B \vee_{\mathscr{K}} C)(0) \supseteq B(0)$ by 1.6 and (9), thus $C(0) \supseteq B(0)$, or $\bigcup C \subseteq \mathfrak{D} + G_0$, thus $\bigcup C \subseteq C(0) + B(0) + \bigcup B \cap \bigcup C = C(0) + \bigcup B \cap \bigcup C \subseteq \bigcup C$ by (11), hence $\bigcup C = C(0) + \bigcup B \cap \bigcup C$. Finally, we have

1) $B(0) \supseteq C(0)$ or 2) $\bigcup B \subseteq \bigcup C$

and simultaneously

a) $C(0) \supseteq B(0)$ or b) $\bigcup C = C(0) + \bigcup B \cap \bigcup C$.

Hence we have one of the following four possibilities:

$1 \wedge a \equiv B(0) = C(0)$. From the above we obtain $B(0) = C(0) = \mathfrak{D}$, hence $B(0) = C(0)$ is an ideal of $\langle \bigcup B, \bigcup C \rangle \Rightarrow (10)$.

$1 \wedge b \Rightarrow B(0) \supseteq C(0)$, $\bigcup C = C(0) + \bigcup B \cap \bigcup C \subseteq B(0) + \bigcup B \cap \bigcup C \subseteq \bigcup B \Rightarrow \bigcup C \subseteq \bigcup B$, $C(0) \subseteq B(0) \Rightarrow C \leqq B \Rightarrow (10)$.

$2 \wedge a \Rightarrow \bigcup B \subseteq \bigcup C$, $B(0) \subseteq C(0) \Rightarrow B \leqq C \Rightarrow (10)$.

$2 \wedge b \Rightarrow \bigcup C = C(0) + \bigcup B \Rightarrow (10)$ provided $B \parallel C$; if not we have (10) again. If we started in (11) from the converse inclusion we should attain the same result (interchanging B and C).

The converse implication. The first part of the condition (10) yields (9) (by 1.6, because both sides of (9) are equal to $(\bigcup B \cup \bigcup C)/B(0)$); from the second part (9) follows trivially.

2.10 Corollary. *The condition*

$$(B \vee_{\mathscr{K}} C) \sqsupset (\bigcup B \cup \bigcup C) = B \vee_P C \neq B \vee_{\mathscr{K}} C$$

implies the commutativity of the congruences B and C.

Proof follows from 2.9, because $B(0) = C(0)$ implies $B(0) \cup C(0) \subseteq \bigcup B \cap \bigcup C$ which is a criterion of commutativity [1] 3.9.

2.11 Theorem. *Put*

$$\mathfrak{B} = \bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)], \quad \mathfrak{C} = \bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)],$$

$$\mathfrak{D} = B \vee_{\mathscr{K}} C(0).$$

Then

$$B \vee_P C = (B \vee_{\mathscr{K}} C) \sqcap (\bigcup B \cup \bigcup C), \tag{12}$$

if and only if (13), (14) *and* (15) *hold, where*

$$\mathfrak{D} \cap (\bigcup B \cup \bigcup C) = B \vee_P C(0), \tag{13}$$

$$(\mathfrak{B} + \mathfrak{D}) \cap \bigcup C = \emptyset, \tag{14}$$

$$(\mathfrak{C} + \mathfrak{D}) \cap \bigcup B = \emptyset. \tag{15}$$

Proof. Let (12) hold. Then (13) holds, too. We shall show (14). If $\mathfrak{B} \neq \emptyset$ then by 1.6, $x \in \mathfrak{B}$ satisfies $B \vee_P C(x) = x + B(0) = B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C) = [(x + \mathfrak{D}) \cap \bigcup B] \cup [(x + \mathfrak{D}) \cap \bigcup C] = [(x + \mathfrak{D}) \cap \bigcup B] \cup [(x + \mathfrak{D}) \cap \bigcup C]$.

Therefore $x + B(0) \supseteq (x + \mathfrak{D}) \cap \bigcup C$. Hence we obtain $(x + \mathfrak{D}) \cap \bigcup C \subseteq [x + B(0)] \cap \bigcup C \subseteq \{\bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)]\} \cap \bigcup C \subseteq (\bigcup B \setminus \bigcup C) \cap \bigcup C = \emptyset$, thus $(x + \mathfrak{D}) \cap \bigcup C = \emptyset$ which is (14).

Analogously, from the supposition $\mathfrak{C} \neq \emptyset$ we obtain (15). Thus, the conditions (13), (14) and (15) are necessary.

Sufficiency. By 1.6 and 1.3 we obtain from (13) the following results:

I. $x \in \bigcup B \cap \bigcup C \Rightarrow B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C) = (x + \mathfrak{D}) \cap (\bigcup B \cup \bigcup C) = x + [\mathfrak{D} \cap (\bigcup B \cup \bigcup C)] = x + B \vee_P C(0) = B \vee_P C(x)$.

The middle equality may be obtained as follows. Evidently $\supseteq$ holds. Conversely, if $x + d \in \bigcup B \cup \bigcup C$ for some $d \in \mathfrak{D}$, then $d \in (-x + \bigcup B) \cup (-x + \bigcup C) = \bigcup B \cup \bigcup C$, thus $d \in \mathfrak{D} \cap (\bigcup B \cup \bigcup C)$.

II. If $x \in \mathfrak{B}$ then by (14), $B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C) = (x + \mathfrak{D}) \cap (\bigcup B \cup \bigcup C) = [(x + \mathfrak{D}) \cap \bigcup B] \cup [(x + \mathfrak{D}) \cap \bigcup C] = (x + \mathfrak{D}) \cap \bigcup B = x + (\mathfrak{D} \cap \bigcup B) \subseteq x + [\mathfrak{D} \cap (\bigcup B \cup \bigcup C)] = x + B \vee_P C(0) = B \vee_P C(x) \subseteq B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C)$. Hence $B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C) = B \vee_P C(x)$.

III. If $x \in \mathfrak{C}$ then we obtain the same result $B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C) = B \vee_P C(x)$ analogously to the above.

2.12 Corollary. *Let $B \vee_{\mathscr{K}} C(0) = B(0) + C(0)$. Then*

$$B \vee_P C = (B \vee_{\mathscr{K}} C) \sqcap (\bigcup B \cup \bigcup C).$$

Note. The condition $B \vee_{\mathscr{K}} C(0) = B(0) + C(0)$ is fulfilled e.g. on Abelian and Hamiltonian groups. For those groups Corollary 2.12, i.e. the identity (12), may be easily proved directly. Denote $\bar{B} = G/B(0)$, $\bar{C} = G/C(0)$. Then $B \vee_P C = (\bar{B} \vee \bar{C}) \sqcap (\bigcup B \cup \bigcup C) = G/(B(0) + C(0)) \sqcap (\bigcup B \cup \bigcup C) = \langle \bigcup B, \bigcup C \rangle / (B(0) + C(0)) \sqcap (\bigcup B \cup \bigcup C) = (B \vee_{\mathscr{K}} C) \sqcap (\bigcup B \cup \bigcup C)$. Only the first identity is not evident. It suffices to prove $\geqq$. Let $x[(\bar{B} \vee \bar{C}) \sqcap (\bigcup B \cup \bigcup C)]\, y$. Then $-x + y \in [B(0) + C(0)] \cap (\bigcup B \cup \bigcup C) = \{[B(0) + C(0)] \cap \bigcup B\} \cup \{[B(0) + C(0)] \cap \bigcup C\} = [B(0) + \bigcup B \cap C(0)] \cup [\bigcup C \cap B(0) + C(0)] = B \vee_P C(0)$. In the proof of Corollary 2.12 we have proved $\mathfrak{B} = \emptyset = \mathfrak{C}$. By 1.6, we have $x(B \vee_P C)\, y$.

Proof of 2.12. Using the notation from the above Theorem we shall show $\mathfrak{B} = \emptyset = \mathfrak{C}$; then the conditions (14) and (15) of Theorem are fulfilled. Indeed, $x \in B$, $y \in (x + \mathfrak{D}) \cap \bigcup C \Rightarrow y = x + b_0 + c_0 = c$ for suitable elements $b_0 \in B(0)$, $c_0 \in C(0)$ and $c \in \bigcup C \Rightarrow \bigcup B \ni x + b_0 = c - c_0 \in \bigcup C \Rightarrow x + b_0 \in \bigcup B \cap \bigcup C \Rightarrow x \in (\bigcup B \cap \bigcup C) - b_0 \subseteq B(0) + (\bigcup B \cap \bigcup C)$, a contradiction.

Analogously, we obtain a contradiction starting from the condition $(x + \mathfrak{D}) \cap \bigcup C \neq \emptyset$ for some $x \in \mathfrak{C}$.

Finally, the condition (13) is fulfilled, too, because $\mathfrak{D} \cap (\bigcup B \cup \bigcup C) = \{[B(0) + C(0)] \cap \bigcup B\} \cup \{[B(0) + C(0)] \cap \bigcup C\} = [B(0) + \bigcup B \cap C(0)] \cup [\bigcup C \cap B(0) + C(0)] = B \vee_P C(0)$ ([1] 3.5.7).

2.13 Note. *Let (12) be true. Then*

$$\left.\begin{aligned} \mathfrak{B} \neq \emptyset &\Rightarrow \mathfrak{D} \cap \bigcup B = B(0) \\ \mathfrak{C} \neq \emptyset &\Rightarrow \mathfrak{D} \cap \bigcup C = C(0) \end{aligned}\right\} \Rightarrow C(0) \cap \bigcup B = B(0) \cap \bigcup C.$$

Proof. For $x \in \mathfrak{B}$ we have $B \vee_P C(x) = x + B(0) = B \vee_{\mathscr{K}} C(x) \cap (\bigcup B \cup \bigcup C) = [(x + \mathfrak{D}) \cap \bigcup B] \cup [(x + \mathfrak{D}) \cap \bigcup C] = [x + (\mathfrak{D} \cap \bigcup B)] \cup [(x + \mathfrak{D}) \cap \bigcup C]$. The last square bracket represents the empty set (by (14)), thus $B(0) \leqq \mathfrak{D} \cap \bigcup B$. Analogously $\mathfrak{C} \neq \emptyset \Rightarrow C(0) = \mathfrak{D} \cap \bigcup C$.

Let $\mathfrak{D} \cap \bigcup B = B(0)$ and $\mathfrak{D} \cap \bigcup C = C(0)$. Then $B(0) \cap \bigcup C = \mathfrak{D} \cap \bigcup B \cap \bigcup C = C(0) \cap \bigcup B$.

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