František Šik
Joins of congruences in $\Omega$-groups

Časopis pro pěstování matematiky, Vol. 106 (1981), No. 3, 299--310

Persistent URL: http://dml.cz/dmlcz/118103

Terms of use:
© Institute of Mathematics AS CR, 1981

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
JOINS OF CONGRUENCES IN $\Omega$-GROUPS

FRANTIŠEK ŠIK, Brno

(Received June 15, 1979)

The symmetric and transitive relations in a set $G$ (also called partitions in $G$) form a complete lattice with respect to the set theoretic inclusion; it is denoted by $P(G)$. If $G$ is a universal algebra, the congruences in $G$, i.e. stable partitions in $G$, also form a complete lattice with respect to the same ordering. This lattice is denoted by $\mathcal{K}(G)$ and is a closed $\land$-subsemilattice of $P(G)$. The joins $\lor_p$ and $\lor_\pi$ in these lattices do not coincide in general — in contrast to the joins in the lattice $\pi(G)$ of all partitions on $G$ (i.e. reflexive partitions in $G$) and in the lattice $\mathcal{C}(G)$ of congruences on $G$ (stable partitions on $G$), it is namely $\lor_p = \lor_\pi = \lor_\mathcal{C}$. Naturally, the $P$-join of any two partitions $B$ and $C$ in an algebra $G$ does not depend on any algebraic structure defined on the set $G$, so even the $\mathcal{C}$-join of congruences on the algebra $G$ does not depend on it. In more detail, what is understood by the notion of independence of the join of congruences on an algebraic structure defined on $G$: Let $B$ and $C$ be congruences on an algebra $G$ with a system of operations $F_1$ and $\mathcal{C}_1$ the lattice of all congruences on $(G, F_1)$. If $F_2$ is another system of operations on the set $G$, $\mathcal{C}_2$ the lattice of all congruences on the algebra $(G, F_2)$ and, $B, C \in \mathcal{C}_2$, then $B \lor_\mathcal{C}_1 C = \lor_\mathcal{C}_2 C$.

We shall be interested in a less restricted problem, namely for $F_1 = 0$, i.e. in searching those pairs $B$ and $C$ of congruences in an $\Omega$-group $G$, $\mathcal{K}$-join of which does not depend on the given algebraic structure defined on the set $G$. Thus we shall investigate properties characterizing pairs $B$ and $C$ of congruences in an $\Omega$-group $G$ with the property $B \lor_p C = B \lor_\mathcal{C} C$, and some related problems. We leave the problem of the stronger independence of joins mentioned above open.

We review some of the notation and theory that is needed. A more detailed information may be found in [1—4], especially as to congruences in algebras, see [1] I.

Given a binary relation $A$ in a set $G$ and $x \in G$ we define $A(x) = \{ y \in G : yAx \}$ and $\bigcup A = \bigcup \{ A(x) : x \in G \} \quad ([1] \, 3.5)$. If $A$ is a symmetric and transitive relation in $G$ (i.e. a partition in $G$) and $A(x) \neq \emptyset$, then the set $A(x)$ is said to be the block of the partition $A$ and the set $\bigcup A$ the domain of the partition $A$.

Let $G$ be an algebra. Then $\mathcal{K}(G)$ is a complete lattice with respect to the ordering by inclusion. For $\{ A_x \} \subseteq \mathcal{K}(G)$ we have $\bigwedge_x A_x = \bigcap A_x$. If $G$ is an $\Omega$-group then the
set of all nonempty congruences in $G$ is a closed sublattice of the lattice $\mathcal{K}(G)$, [1] 1.1. Let $A$ be a (nonempty) congruence in $G$. Then $\bigcup A$ is an $\Omega$-subgroup of $G$, $A(0)$ an ideal of $\bigcup A$ and $A = \bigcup A/A(0)$, [1] 1.4. If $\{A_s\} \subseteq \mathcal{K}(G)$ then $\bigcup (\bigvee_s A_s) = = \langle \bigcup A_s \rangle$ and $(\bigvee_s A_s) (0) = \langle \bigcup A_s(0) \rangle_H$, where $\mathfrak{U} = \langle \bigcup A_s \rangle$ is the $\Omega$-subgroup generated by the set $\bigcup A_s$ and $\langle \bigcup A_s(0) \rangle_H$ is the ideal generated in $\mathfrak{U}$ by the set $\bigcup A_s(0)$, [1] 1.6.

The results of the present paper are based on Lemma 1.6, in which a description of blocks of the partition $B \vee_p C$ is given, where $B$ and $C$ are congruences in $G$ and $G$ is an $\Omega$-group. Further, criteria are given for the validity of the following identities:

\[(B \vee_p C) \sqcup (\bigcup B \cap \bigcup C) = (B \vee C) \sqcup (\bigcup B \cap \bigcup C)\] (Theorem 2.3),

\[B \vee_p C = B \vee C\] (Theorems 2.5 and 2.7),

\[(B \vee C) \sqcup (\bigcup B \cap \bigcup C) = B \vee_p C\] (Theorem 2.9),

\[(B \vee C) \sqcup (\bigcup B \cap \bigcup C) = B \vee_p C\] (Theorems 2.11 and 2.12),

where the partition $A \sqcup \mathfrak{U}$ (or $\mathfrak{U} \sqsubset A$), called the closure of the subset $\mathfrak{U}(\subseteq G)$ in the partition $A(A \in P(G))$, is the set of all blocks of $A$ that are incident with $\mathfrak{U}$ and $A \cap \mathfrak{U} (= \mathfrak{U} \cap A) = \{A^1 \cap \mathfrak{U} : A^1 \in A, A^1 \cap \mathfrak{U} \neq \emptyset\}$ (called the intersection of the partition $A$ and the subset $\mathfrak{U}$) — see [4] 2.3.

In what follows $G$ will denote an $\Omega$-group and $B$ and $C$ (nonempty) congruences in $G$, unless otherwise indicated.

1.1 Lemma. If $x \in \bigcup B \cap \bigcup C$ then $BC(x) = x + BC(0) = BC(0) + x$.

Proof. For $x \in \bigcup B \cap \bigcup C$ we have

\[y \in BC(0) + x \iff y - x \in BC(0) \iff \exists a \in G, (y - x)BaC0 \iff \exists a \in G, yB(a + x)Cx \iff yBCx \iff y \in BC(x).\]

Similarly $y \in x + BC(0) \iff y \in BC(x)$.

1.2 Lemma. If $x \in \bigcup B \cap \bigcup C$ then $BCB \ldots (x) = BC(x)$, where the product on the left-hand side contains a finite number ($\geq 2$) of factors.

Proof by induction on the number $n$ of factors. It suffices to show $BCB \ldots (x) \subseteq BC(x)$ for $x \in \bigcup B \cap \bigcup C$, because the converse inclusion is evident. In fact, if $x \in \bigcup B \cap \bigcup C$ then $yBCx = yBCxBxCx \ldots x = y(BCB \ldots )x$.

The inclusion $\subseteq$ is valid for $n = 2$. First, we shall prove it for $n = 3$. If $x \in \bigcup B \cap \bigcup \cap \bigcup C$ then

\[y \in BCB(x) \iff \exists a \in G, yBCaBx \iff (1.1) \exists a \in G, y \in a + BC(0), a \in x + B(0) \iff \iff y \in x + B(0) + BC(0) = = x + B(0) + B(0) + \bigcup B \cap C(0) = x + BC(0)\]
(by [1] 3.5.5). By 1.1, the last expression is equal to $BC(x)$. The inductive hypothesis: Let $n \geq 4$ and let $x \in \bigcup B \cap \bigcup C$ imply $BCB \ldots (x) \subseteq BC(x)$, whenever the number $p$ of factors on the left-hand side fulfils $2 < p \leq n - 1$. Now, let $x \in \bigcup B \cap \bigcup C$, $yBCB \ldots x$ and let the product contain $n \geq 4$ factors. Then there exists $a \in \bigcup B \cap \bigcup C$ such that $yBCa(BC \ldots) x$, thus by 1.1 $y - a \in BC(0)$. By assumption $a \in BCB \ldots (x) \subseteq BC(x)$, hence $y = (y - a) + a \in BC(0) + BC(x) \subseteq BC(x)$. The last inclusion follows from the implication $tBC0, zBCx \Rightarrow (t + z) BCx$. So we have got that $x \in \bigcup B \cap \bigcup C$ satisfies $BCB \ldots (x) \subseteq BC(x)$, which was to be proved.

1.3 Lemma. If $x \in \bigcup B \cap \bigcup C$ then

$$B \lor_p C(0) = B(0) \cup BC(0) \cup C(0) \cup CB(0) =$$

$$= [B(0) + \bigcup B \cap C(0)] \cup [C(0) + \bigcup C \cap B(0)],$$

$$B \lor_p C(x) = B(x) \cup BC(x) \cup C(x) \cup CB(x) =$$

$$= x + B \lor_p C(0) = B \lor_p C(0) + x.$$  

The member in the first square bracket or in the second one is an ideal of the $\Omega$-group $B(0) + \bigcup B \cap C$ or $C(0) + \bigcup B \cap C$, respectively. The order of summands (in one or both the square brackets) may be changed.

Proof. The first assertion is Corollary 3.5.7 [1]. Proof of the second one follows by a similar argument: Denote $B_n = BCB \ldots$ and $C_n = CBC \ldots$, provided the product on the right-hand sides contains $n (\geq 1)$ factors. Now, the assertion follows from 1.2 and 1.1 because

$$B \lor_p C(x) = \bigcup_{n=1}^\infty B_n(x) \cup \bigcup_{n=1}^\infty C_n(x) = B(x) \cup BC(x) \cup C(x) \cup CB(x) =$$

$$= [x + B(0)] \cup [x + BC(0)] \cup [x + C(0)] \cup [x + CB(0)] =$$

$$= x + [B(0) \cup C(0) \cup BC(0) \cup CB(0)] = x + B \lor_p C(0).$$

Analogously we obtain the identity $B \lor_p C(x) = B \lor_p C(0) + x$.

1.4 A generalization of the first assertion of Lemma 1.3 for an arbitrary number of congruences will be given in the following

Theorem. Let $B_\alpha (\alpha \in A)$ be congruences in $G$. Then

$$(\bigvee_{\alpha \in A} B_\alpha)(0) = \bigcup_{n=1}^\infty \bigcup_{\alpha_1, \ldots, \alpha_n} W(\alpha_1, \ldots, \alpha_n),$$

where $W(\alpha_1) = B_{\alpha_1}(0)$, $W(\alpha_1, \ldots, \alpha_n) = W(\alpha_1, \ldots, \alpha_{n-1}) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0)$, $n = 2, 3, \ldots$ and $\alpha_1, \ldots, \alpha_n$ is an $n$-tuple of elements of $A$. 301
Note. In the definition of $W(\alpha_1, \ldots, \alpha_n)$ it is possible to interchange both the summands (because $B_{\alpha_1}(0)$ is an ideal of $\bigcup B_{\alpha_n}$).

Proof. Denote $V = \bigvee_{\alpha \in A} B_{\alpha}$ and let $W$ stands for the expression on the right-hand side of the required identity. Let $y \in V(0)$. Then $0 B_{\alpha_1} y_1 B_{\alpha_2} y_2 \cdots y_{n-1} B_{\alpha_{n-1}} y_n B_{\alpha_n} y$ for suitable $y_1, \ldots, y_n \in G$ and $\alpha_1, \ldots, \alpha_n \in A$.

Hence
\[
y_1 \in B_{\alpha_1}(0) = W(\alpha_1), \quad y_2 \in y_1 + B_{\alpha_2}(0) \subseteq B_{\alpha_2}(0) \cap \bigcup B_{\alpha_2} + B_{\alpha_3}(0) = W(\alpha_1) \cap \bigcup B_{\alpha_2} + B_{\alpha_3}(0) = W(\alpha_1, \alpha_2), \ldots, y_i = y_{i-1} + B_{\alpha_i}(0) \subseteq W(\alpha_1, \ldots, \alpha_{i-1}) \cap \bigcup B_{\alpha_i} + B_{\alpha_i}(0) = W(\alpha_1, \ldots, \alpha_i), \ldots, y \in y_n + B_{\alpha_n}(0) \subseteq W(\alpha_1, \ldots, \alpha_{n-1}) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) = W(\alpha_1, \ldots, \alpha_n).
\]

Therefore $V(0) \subseteq W$. Now let $\alpha_1, \ldots, \alpha_n$ be an arbitrary $n$-tuple of elements of $A$. Then $W(\alpha_1, \ldots, \alpha_n) \subseteq V(0)$. In fact, if $n = 1$ then $W(\alpha_1) = B_{\alpha_1}(0) \subseteq V(0)$. We use induction on $n$. By the inductive hypothesis $W(\alpha_1, \ldots, \alpha_{n-1}) \subseteq V(0)$ we have
\[
W(\alpha_1, \ldots, \alpha_n) = W(\alpha_1, \ldots, \alpha_{n-1}) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) \subseteq V(0) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) \subseteq V(0).
\]

We obtain the last inclusion as follows. For $v \in V(0) \cap \bigcup B_{\alpha_n}$ the block $v + B_{\alpha_n}(0)$ of the partition $B_{\alpha_n}$ meets $V(0)$, so $v + B_{\alpha_n}(0) \subseteq V(0)$ for all $v \in V(0) \cap \bigcup B_{\alpha_n}$ and thus
\[
V(0) \cap \bigcup B_{\alpha_n} + B_{\alpha_n}(0) \subseteq V(0), \quad W(\alpha_1, \ldots, \alpha_n) \subseteq V(0) \quad \text{and} \quad W \subseteq V(0).
\]

Finally $V(0) \subseteq W \subseteq V(0)$, hence $V(0) = W$.

We can obtain the first assertion of Lemma 1.3 as a special case of Theorem 1.4 in the following way. Denote $B = B_1$ and $C = B_2$. Evidently $W(1) = B_1(0) \subseteq B_2(0) \cap \bigcup B_1 + B_1(0) = W(2, 1)$; analogously $W(2) \subseteq W(1, 2)$. Further $W(1, 2, 1) = [B_1(0) \cap \bigcup B_2 + B_2(0)] \cap \bigcup B_1 = [B_2(0) \cap \bigcup B_2 + B_2(0) + B_1(0)] \cap \bigcup B_1 = B_2(0) + B_1(0) \cap \bigcup B_1 = B_2(0) \cap \bigcup B_1 + B_1(0) = W(2, 1)$.

Similarly $W(2, 1, 2) = W(1, 2), \ W(1, 1) = W(1)$ and $W(2, 2) = W(2)$. Iterating this procedure we obtain $V(0) = \bigcup_{n=1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_n} W(\alpha_1, \ldots, \alpha_n) = W(1, 2) \cup W(2, 1)$ which is the required assertion.

1.5 In the next theorem, another construction of the set $(\bigvee_{\alpha \in A} B_{\alpha})(0)$ is given.

Theorem. Let $B_{\alpha}$ ($\alpha \in A$) be congruences in $G$. Then
\[
(\bigvee_{\alpha \in A} B_{\alpha})(0) = \bigcup_{\beta \in A} \{ \bigcup B_{\alpha} \cap [\bigcup B_{\beta}(0)] + B_{\beta}(0) \}.
\]
Note. It is possible to interchange the summands on the right-hand side (because $B_a(0)$ is an ideal of $\cup_{\beta \in \mathcal{A}}^{\beta + x} B_a$). Again, on the right-hand side, $\cup$ can be put in place of $\cup$. The symbol $[\mathcal{A}]$ denotes the subgroup of $G$ generated by the subset $\mathcal{A}$ of $G$.

Proof. Denote $V = \bigvee_{\beta \in \mathcal{A}} B_a$. We have

$$\bigcup_{\beta \in \mathcal{A}} \{ \bigcup_{\alpha = 1}^{\infty} \bigcup_{n = 1}^{\infty} \bigcup_{\alpha_1, \ldots, \alpha_n} B_{\alpha_1}(0) + \ldots + B_{\alpha_n}(0) \} + B_a(0) =$$

$$= \bigcup_{\beta \in \mathcal{A}} \bigcup_{\alpha_1, \ldots, \alpha_n} \{ \bigcup_{\beta \in \mathcal{A}} B_{\alpha_1}(0), \ldots, B_{\alpha_n}(0) \} + B_a(0)$$

where $\alpha_1, \ldots, \alpha_n$ runs through all n-tuples of elements of $A$. We shall show that

$$\bigcup_{\alpha_1, \ldots, \alpha_n} \{ \bigcup_{\beta \in \mathcal{A}} B_{\alpha_1}(0), \ldots, B_{\alpha_n}(0) \} + B_a(0) \subseteq V(0),$$

and so the inclusion $\supseteq$ in the assertion of Theorem will be proved.

Thus, let $b_1, \ldots, b_k$ be arbitrary elements of the set $B_{\alpha_1}(0) \cup \ldots \cup B_{\alpha_n}(0)$ with $b_k + \ldots + b_1 \in \bigcup B_a$ and let $b \in B_a$. If $k = 1$ then $b_1 + b \in V(0)$, since the block $b_1 + B_a(0)$ of the partition $B_a$ meets $V(0)$ and $B_a \subseteq V$. We use induction on $k$. Suppose that $b_1, \ldots, b_{k+1} \in B_{\alpha_1}(0) \cup \ldots \cup B_{\alpha_n}(0)$, $b_{k+1} + \ldots + b_1 \in \bigcup B_a$, $b \in B_a(0)$ and $b_k + \ldots + b_1 + b \in V(0)$. Then $b_{k+1} + \ldots + b_1 + b \in B_{\alpha_1}(0) + B_a(0) + V(0)$ for some $\alpha_{k+1}$. Since for an arbitrary $v \in V(0)$ the block $B_{\alpha_1}(0) + v$ of the partition $B_{\alpha_1}$ meets $V(0)$ then $B_{\alpha_1}(0) + v \subseteq V(0)$, whence $B_{\alpha_1}(0) + V(0) \subseteq V(0)$. Finally $b_{k+1} + \ldots + b_1 + b \in V(0)$ which completes the proof by induction.

1.6 In the next lemma the description of blocks of the partition $B \vee_p C$ is given. This is the crucial lemma for our study.

Lemma. The following implications hold:

1. $x \in \bigcup B \cap \bigcup C \Rightarrow B \vee_p C(x) = x + B \vee_p C(0) = B \vee_p C(0) + x$,

2. $x \in \bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)] \Rightarrow B \vee_p C(x) = x + B(0) = B(0) + x = B(x)$,

3. $x \in \bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)] \Rightarrow B \vee_p C(x) = x + C(0) = C(0) + x = C(x)$.

The blocks (1) are exactly the blocks of the partition $(B \vee_p C) \supseteq \bigcup B \cap \bigcup C$, the domain of which is $(B(0) \cup C(0)) + (\bigcup B \cap \bigcup C)$; the blocks (2) and (3) are the remaining blocks of the partition $B \vee_p C$. The blocks (2) cover the set $\bigcup B \setminus [B(0) + + (\bigcup B \cap \bigcup C)]$, and the blocks (3) cover the set $\bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)]$. 

303
Proof. (1) follows from 1.3. Thus the system of sets \( \{B \lor_P C(0) + x : x \in \bigcup B \cap \bigcup C\} \) is equal to the set of the blocks of the partition \( (B \lor_P C) \sqcup (\bigcup B \cap \bigcup C) \).

The domain \( \mathfrak{X} \) of this partition is \( \mathfrak{X} = B \lor_P C(0) + (\bigcup B \cap \bigcup C) = \{[B(0) + \bigcup B \cap C(0)] \cup [C(0) + \bigcup C \cap B(0)]\} + (\bigcup B \cap \bigcup C) = (B(0) \cup C(0)) + (\bigcup B \cap \bigcup C). \)

If \( x \in \bigcup B \setminus \mathfrak{X} \), then \( B \lor_P C(x) = B(x) \) and if \( x \in \bigcup C \setminus \mathfrak{X} \), then \( B \lor_P C(x) = C(x) \).

Finally let us recall that \( \bigcup B \setminus \mathfrak{X} = \bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)] \), as \( [C(0) + (\bigcup B \cap \bigcup C)] \cap \bigcup B = \bigcup B \cap \bigcup C. \) Analogously \( \bigcup C \setminus \mathfrak{X} = \bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)]. \)

2.1 Definition. \( \langle \bigcup B, \bigcup C \rangle \) is the \( \Omega \)-subgroup generated in \( G \) by the set \( \bigcup B \cup \bigcup C \) and \( \langle B(0), C(0) \rangle \) is the ideal generated in \( \mathfrak{A} = \langle \bigcup B, \bigcup C \rangle \) by the set \( B(0) \cup C(0). \)

2.2 Lemma. Let \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \) be subgroups of a group \( G \). If \( \mathfrak{A} \cup \mathfrak{B} = \mathfrak{C} \), then the sets \( \mathfrak{A} \) and \( \mathfrak{B} \) are comparable by inclusion.

Proof. If the sets \( \mathfrak{A} \) and \( \mathfrak{B} \) are incomparable by inclusion, then there exists elements \( x \in \mathfrak{A} \setminus \mathfrak{B} \) and \( y \in \mathfrak{B} \setminus \mathfrak{A} \) and it holds \( C \ni x + y \in \mathfrak{A} \cup \mathfrak{B} = \mathfrak{C} \), a contradiction.

2.3 Theorem. The identity

\[
(B \lor_P C) \sqcup (\bigcup B \cap \bigcup C) = (B \lor_C^\mathfrak{X} C) \sqcup (\bigcup B \cap \bigcup C)
\]

holds if and only if \( B(0) \subseteq \bigcup C \) or \( C(0) \subseteq \bigcup B \), and simultaneously \( B(0) + C(0) \) is an ideal of \( \langle \bigcup B, \bigcup C \rangle \).

Proof. The condition (1) is equivalent to the following one:

\[
B \lor_P C(x) = B \lor_C^\mathfrak{X} C(x) \quad \text{for each} \quad x \in \bigcup B \cap \bigcup C.
\]

By 1.3, if \( x \in \bigcup B \cap \bigcup C \) then

\[
B \lor_P C(x) = x + B \lor_P C(0) =
\]

\[
x + ([B(0) + \bigcup B \cap C(0)] \cup [\bigcup C \cap B(0) + C(0)])
\]

and further

\[
B \lor_C^\mathfrak{X} C(x) = x + \langle B(0), C(0) \rangle_{\mathfrak{A}}, \quad \text{where} \quad \mathfrak{A} = \langle \bigcup B, \bigcup C \rangle \quad ([1] \text{ 1.6}).
\]

Then the identity \( B \lor_P C(x) = B \lor_C^\mathfrak{X} C(x) \) is true for each \( x \in \bigcup B \cap \bigcup C \) if and only if the following identity (2) is valid:

\[
[B(0) + \bigcup B \cap C(0)] \cup [\bigcup C \cap B(0) + C(0)] = \langle B(0), C(0) \rangle_{\mathfrak{A}}.
\]

304
Let (2) hold. The left-hand side of (2) is the union of two \( \Omega \)-subgroups. Denote them by \( \mathfrak{A} \) and \( \mathfrak{B} \). The right-hand side is an \( \Omega \)-subgroup. By 2.2 the sets \( \mathfrak{A} \) and \( \mathfrak{B} \) are comparable. Thus we have e.g.

\[
B(0) + \bigcup B \cap C(0) \subseteq \bigcup C \cap B(0) + C(0). 
\]

The right-hand side is a subset of \( \bigcup C \), hence \( B(0) \subseteq \bigcup C \). Now, (2) has the form

\[
B(0) + C(0) = \langle B(0), C(0) \rangle_{\mathfrak{A}}. 
\]

It follows that \( B(0) + C(0) \) is an ideal of \( \langle \bigcup B, \bigcup C \rangle \).

If we start from the converse inclusion in (3) we obtain \( C(0) \subseteq \bigcup B \) and (4).

To prove the converse implication it suffices to verify that (2) is true whenever the conditions of Theorem are fulfilled. The left-hand side of (2) is equal to \( B(0) + C(0) \), and by supposition, this is an ideal of \( \langle \bigcup B, \bigcup C \rangle \), hence \( B(0) + C(0) = \langle B(0), C(0) \rangle_{\mathfrak{A}} \). This completes the proof.

2.4 Corollary. If \( B \) and \( C \) are congruences on \( G \) then \( B \vee_P C = B \vee \chi C \) (\( = B \vee_P \chi C \)).

2.5 If we investigate conditions which guarantee the validity of the identity \( B \vee_P C = B \vee \chi C \) for congruences \( B \) and \( C \) in \( G \), we may restrict ourselves to incomparable congruences, because comparable congruences fulfil it evidently.

Theorem. If \( B \) and \( C \) are incomparable congruences in \( G \) then

\[
B \vee_P C = B \vee \chi C
\]

if and only if

\[
B(0) + \bigcup C = \bigcup B \quad \text{or} \quad \bigcup B + C(0) = \bigcup C
\]

or equivalently if

\[
\bigcup (B \vee_P C) = \bigcup (BC) \quad \text{or} \quad \bigcup (CB)
\]

or equivalently if

\[
\bigcup (B \vee \chi C) = \bigcup (BC) \quad \text{or} \quad \bigcup (CB).
\]

Note. Due to the symmetry between \( B \) and \( C \) in (5) the summands in (6) can be interchanged.

Proof. \( 5 \Rightarrow 6 \). Because \( \bigcup B \cup \bigcup C = \langle \bigcup B, \bigcup C \rangle \), 2.2 implies that either \( \bigcup B \supseteq \bigcup C \) or \( \bigcup B \subseteq \bigcup C \), say \( \bigcup B \supseteq \bigcup C \). Then we have \( B(0) + \bigcup C \subseteq \bigcup B \). If \( + \) then there exists \( x \in \bigcup B \setminus [B(0) + \bigcup C] \) had by 1.6, this \( x \) satisfies \( B \vee_P C(x) = B(x) = x + B(0) \). Since \( B \vee \chi C(x) = x + \langle B(0), C(0) \rangle_{B(0)}, (5) \) implies \( B(0) = \langle B(0), C(0) \rangle_{B(0)}, \)
thus $B(0) \supseteq C(0)$ and finally $B \supseteq C$, a contradiction. Analogously, if $\cup C \supseteq \cup B$, then $\cup B + C(0) = \cup C$.

$6 \Rightarrow 5$. Let $B(0) + \cup C = \cup B$ be true. We shall prove that $B(0) + C(0)$ is an ideal of $\cup B = \langle \cup B, \cap \rangle$. The proof is based on the elementary procedures to follow. Denote by $b$ or $b'$ (with indices if necessary) elements of $\cup B$ or $B(0)$, respectively. Similarly for $\cup C$ and $C(0)$. The set $B(0) + C(0)$ is an $\Omega$-subgroup (since $C(0) \subseteq \subseteq \subseteq \cup B$). We shall show it is normal in $\cup B$. Arbitrary elements $b, b', c, c'$ satisfy $b + b' + c = b' + c + b + c - b - b' = b' + b'' + c + c - c - b' = b' + b'' + b'' + b'' + c' \in B(0) + C(0)$.

If $\omega$ is an $n$-ary operation in $G$ we shall shortly write $\omega_{\ast \omega}$ instead of $g_{\ast \omega}$.

So we have shown that $B(0) + C(0)$ is an ideal of $\cup B$.

By 2.3, $(B \vee_p C) \sqsubseteq \cup C = (B \vee_\& C) \sqsubseteq \cup C$ is true. By 1.6, $(B \vee_p C) \sqsubseteq \cup C = B \vee_p C$ holds and the identity $B(0) + \cup C = \cup B$ yields $(B \vee_\& C) \sqsubseteq \cup C = B \vee_\& C$. This completes the proof of $6 \Rightarrow 5$. The remaining part of the assertion follows from [1] 3.7.5.

2.6 Corollary. ([1] 3.11) If $\cup B = \cup C$ then $B \vee_p C = B \vee_\& C$.

Proof follows from 2.5 since $B(0) \subseteq \cup B = \cup C$ implies $B(0) + \cup C = \cup B$.

The converse implication is true for commuting congruences.

2.7 Corollary. If $B$ and $C$ commute and $B \parallel C$ then $B \vee_p C = B \vee_\& C$ if and only if $\cup B = \cup C$.

Proof. $\Rightarrow$: If $B$ and $C$ commute then [1] 3.9 yields $B(0) \cup C(0) \subseteq \cup B \cap \cup C$ and by 2.5 the condition (6) is fulfilled. This condition gives $\cup B = \cup C$.

The converse follows from 2.6.

2.8 Proposition. Let $G$ be an $\Omega$-group. Then the following conditions are equivalent:

(a) The lattice $\mathcal{X}(G)$ is a sublattice of the lattice $P(G)$.
(b) $\mathcal{X}(G)$ is a chain.
(c) $\mathcal{X}(G)$ has three elements only, $G/G, G/\{0\}$ and $\{0\}/\{0\}$.
(e) $G$ has no proper $\Omega$-subgroups.
Note. If $G$ is a group then the condition (e) reads: $G$ is a cyclic group of prime order.

Proof. $a \Rightarrow d$. Let $\mathcal{C}$ be a proper $\Omega$-subgroup of $G$, $B = G/\{0\}$ and $C$ an arbitrary congruence in $G$ with $\cup C = \mathcal{C}$. If $C(0) \neq \{0\}$ then $B$ and $C$ are incomparable, thus $\mathcal{C} = G$ by 2.5, a contradiction. Hence $C(0) = \{0\}$. In particular, for $C = \mathcal{C}/\mathcal{C}$ we have $C(0) = \mathcal{C} = \{0\}$, a contradiction. Therefore $G$ has no proper $\Omega$-subgroups.

d \Rightarrow c \Rightarrow b \Rightarrow a$ is evident.

2.9 Theorem. The identity

$$\tag{9} (B \lor C) \sqsupset (\cup B \cup \cup C) = B \lor \lor C$$

holds if and only if

$$\tag{10} B(0) = C(0) \text{ is an ideal of } \langle \cup B, \cup C \rangle \text{ or } B \lor C = B \lor \lor C.$$

Note. The condition (9) reads that the set of all blocks of the partition $B \lor C$ is a subset of the set of all blocks of the partition $B \lor C$. These blocks of the partition $B \lor C$ cover the domain $\cup B \cup \cup C$ of the partition $B \lor C$.

Proof. Denote $\mathcal{D} = (B \lor C)(0)$ and suppose (9). By 1.3, $\mathcal{D} = (B \lor C)(0) = (B \lor C)(0) = [B(0) + \cup B \cap C(0)] \cup [C(0) + \cup C \cap B(0)] \subseteq B(0) + C(0) \subseteq \mathcal{D}$, thus $\mathcal{D} = B(0) + C(0) = [B(0) + \cup B \cap C(0)] \cup [C(0) + \cup C \cap B(0)]$. The left-hand side is a subgroup, the right-hand side is the union of two subgroups. By 2.2 we have e.g.

$$\tag{11} B(0) + \cup B \cap C(0) \subseteq C(0) + \cup C \cap B(0).$$

The right-hand side is contained in $\cup C$, hence $B(0) \subseteq \cup C$. Denote $G_0 = \cup B \cap \cup C$. Then either $\cup B \setminus (\mathcal{D} + G_0) = 0$, hence $B(0) = (B \lor C)(0) \supseteq C(0)$ by 1.6 and (9), hence $B(0) \supseteq C(0)$, or $\cup B \subseteq \mathcal{D} + G_0$, thus $\cup B \subseteq C(0) + B(0) + \cup B \cap \cup C = C(0) + \cup B \cap \cup C \subseteq \cup C$ by (11). Hence $\cup B \subseteq \cup C$.

Simultaneously either $\cup C \setminus (\mathcal{D} + G_0) = 0$, then $C(0) = (B \lor C)(0) \supseteq B(0)$ by 1.6 and (9), thus $C(0) \supseteq B(0)$, or $\cup C \subseteq \mathcal{D} + G_0$, thus $\cup C \subseteq C(0) + B(0) + \cup B \cap \cup C = C(0) + \cup B \cap \cup C \subseteq \cup C$ by (11), hence $\cup C = C(0) + \cup B \cap \cup C$.

Finally, we have

1) $B(0) \supseteq C(0)$ or 2) $\cup B \subseteq \cup C$

and simultaneously

a) $C(0) \supseteq B(0)$ or b) $\cup C = C(0) + \cup B \cap \cup C$.

Hence we have one of the following four possibilities:

1 $\Rightarrow a \Rightarrow B(0) = C(0)$. From the above we obtain $B(0) = C(0) = \mathcal{D}$, hence $B(0) = C(0)$ is an ideal of $\langle \cup B, \cup C \rangle \Rightarrow (10)$.

1 $\Rightarrow b \Rightarrow B(0) \supseteq C(0)$, $\cup C = C(0) + \cup B \cap \cup C \subseteq B(0) + \cup B \cap \cup C \subseteq \cup B \Rightarrow \cup C \subseteq \cup B, C(0) \subseteq B(0) \Rightarrow C \subseteq B \Rightarrow (10)$.

2 $\Rightarrow a \Rightarrow \cup B \subseteq \cup C, B(0) \subseteq C(0) \Rightarrow B \subseteq C \Rightarrow (10)$. 

307
2 \land b \Rightarrow \bigcup C = C(0) + \bigcup B \Rightarrow (10) provided B \parallel C; if not we have (10) again. If we started in (11) from the converse inclusion we should attain the same result (interchanging B and C).

The converse implication. The first part of the condition (10) yields (9) (by 1.6, because both sides of (9) are equal to \((\bigcup B \cup \bigcup C)/B(0))\); from the second part (9) follows trivially.

2.10 Corollary. The condition

\[(B \lor x \ C) \sqsupset (\bigcup B \cup \bigcup C) = B \lor_p C \neq B \lor x \ C\]

implies the commutativity of the congruences B and C.

Proof follows from 2.9, because B(0) = C(0) implies B(0) \cup C(0) \subseteq \bigcup B \cap \bigcup C which is a criterion of commutativity [1] 3.9.

2.11 Theorem. Put

\[\mathfrak{B} = \bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)], \quad \mathfrak{C} = \bigcup C \setminus [C(0) + (\bigcup B \cap \bigcup C)]\]

\[\mathfrak{D} = B \lor x \ C(0)\]

Then

(12) \[B \lor_p C = (B \lor x \ C) \sqcap (\bigcup B \cup \bigcup C),\]

if and only if (13), (14) and (15) hold, where

(13) \[\mathfrak{D} \cap (\bigcup B \cup \bigcup C) = B \lor_p C(0),\]

(14) \[(\mathfrak{B} + \mathfrak{D}) \cap \bigcup C = \emptyset,\]

(15) \[(\mathfrak{C} + \mathfrak{D}) \cap \bigcup B = \emptyset.\]

Proof. Let (12) hold. Then (13) holds, too. We shall show (14). If \(\mathfrak{B} \neq \emptyset\) then by 1.6, \(x \in \mathfrak{B}\) satisfies \(B \lor_p C(x) = x \lor B(0) = B \lor x \ C(x) \cap (\bigcup B \cup \bigcup C) = [(x + \mathfrak{D}) \cap \bigcup B] \cup [(x + \mathfrak{D}) \cap \bigcup C] = [(x + \mathfrak{D}) \cap \bigcup B] \cup [(x + \mathfrak{D}) \cap \bigcup C].\)

Therefore \(x + B(0) \supseteq (x + \mathfrak{D}) \cap \bigcup C\). Hence we obtain \((x + \mathfrak{D}) \cap \bigcup C \subseteq \left\{x + B(0)\right\} \cap \bigcup C \subseteq \left\{\bigcup B \setminus [B(0) + (\bigcup B \cap \bigcup C)]\right\} \cap \bigcup C \subseteq (\bigcup B \cap \bigcup C) \cap \bigcup C = \emptyset\), thus \((x + \mathfrak{D}) \cap \bigcup C = \emptyset\) which is (14).

Analogously, from the supposition \(\mathfrak{C} \neq \emptyset\) we obtain (15). Thus, the conditions (13), (14) and (15) are necessary.

Sufficiency. By 1.6 and 1.3 we obtain from (13) the following results:

I. \(x \in \bigcup B \cap \bigcup C \Rightarrow B \lor x \ C(x) \cap (\bigcup B \cup \bigcup C) = (x + \mathfrak{D}) \cap (\bigcup B \cup \bigcup C) = x + [\mathfrak{D} \cap (\bigcup B \cup \bigcup C)] = x + B \lor_p C(0) = B \lor_p C(x).\)

The middle equality may be obtained as follows. Evidently \(\supseteq\) holds. Conversely, if \(x + d \in \bigcup B \cup \bigcup C\) for some \(d \in \mathfrak{D}\), then \(d \in (-x + \bigcup B) \cup (-x + \bigcup C) = \bigcup B \cup \bigcup C,\) thus \(d \in \mathfrak{D} \cap (\bigcup B \cup \bigcup C).\)
II. If \( x \in B \) then by (14), \( B \vee^\infty C(x) \cap (\cup B \cup C) = (x + D) \cap (\cup B \cup C) = \) 
\( [x + D] \cap \cup B \cup [(x + D) \cap C] = (x + D) \cap B = x + (D \cap B) \subseteq x + \) 
\( D \cap (\cup B \cup C) = x + B \vee^p C(0) = B \vee^p C(x) \subseteq B \vee^\infty C(x) \cap (\cup B \cup C). \)

Hence \( B \vee^\infty C(x) \cap (\cup B \cup C) = B \vee^p C(x). \)

III. If \( x \in C \) then we obtain the same result \( B \vee^\infty C(x) \cap (\cup B \cup C) = \) 
\( = B \vee^p C(x) \) analogously to the above.

2.12 Corollary. Let \( B \vee^\infty C(0) = B(0) + C(0). \) Then

\[ B \vee^\infty C(0) = (B \vee^\infty C) \cap (\cup B \cup C). \]

Note. The condition \( B \vee^\infty C(0) = B(0) + C(0) \) is fulfilled e.g. on Abelian and Hamiltonian groups. For those groups Corollary 2.12, i.e. the identity (12), may be easily proved directly. Denote \( \bar{B} = G/B(0), \bar{C} = G/C(0). \) Then \( B \vee^p C = (\bar{B} \vee \bar{C}) \cap \) 
\( \cap (\cup B \cup \cup C) = G/[(B(0) + C(0)) \cap (\cup B \cup \cup C)] = \langle B(0) + C(0) \rangle \cap (\cup B \cup \cup C) = B(0) + \cup B \cap C(0) + (C(0)) \cap \cup C = \) 
\( = B(0) + \cup B \cap C(0) \cup [\cup C \cap B(0) + C(0)] = B \vee^p C(0). \) In the proof of Corollary 2.12 we have proved \( \mathfrak{B} = \emptyset = \mathfrak{C}. \) By 1.6, we have \( x(B \vee^p C) y. \)

Proof of 2.12. Using the notation from the above Theorem we shall show \( \mathfrak{B} = \) 
\( = \emptyset = \mathfrak{C}; \) then the conditions (14) and (15) of Theorem are fulfilled. Indeed, \( x \in B, \) 
\( y \in (x + D) \cap \cup C \Rightarrow y = x + b_0 + c_0 = c \) for suitable elements \( b_0 \in B(0), \ c_0 \in \) 
\( \in C(0) \) and \( c \in \cup C \Rightarrow \cup B \ni x + b_0 = c - c_0 \in \cup C \Rightarrow x + b_0 \in \cup B \cap \cup C \Rightarrow x \in \) 
\( \in (\cup B \cap \cup C) - b_0 \subseteq B(0) + (\cup B \cap \cup C), \) a contradiction.

Analogously, we obtain a contradiction starting from the condition \( (x + D) \cap \) 
\( \cap \cup C \neq \emptyset \) for some \( x \in \mathfrak{C}. \)

Finally, the condition (13) is fulfilled, too, because \( D \cap (\cup B \cup \cup C) = \) 
\( = [[B(0) + + C(0)] \cap \cup B] \cup [[B(0) + C(0)] \cap \cup C] = \) 
\( = [B(0) + \cup B \cap C(0)] \cup [\cup C \cap B(0) + + C(0)] = B \vee^p C(0) \) (11.3.5.7).

2.13 Note. Let (12) be true. Then

\[ \mathfrak{B} \neq \emptyset \Rightarrow D \cap \cup B = B(0) \) 
\[ \mathfrak{C} \neq \emptyset \Rightarrow D \cap \cup C = C(0). \]

Proof. For \( x \in \mathfrak{B} \) we have \( B \vee^p C(x) = x + B(0) = B \vee^\infty C(x) \cap (\cup B \cup \cup C) = \) 
\( = [(x + D) \cap \cup B] \cup [(x + D) \cap \cup C] = [x + (D \cap \cup B)] \cup [(x + D) \cap \cup C]. \)

The last square bracket represents the empty set (by (14)), thus \( B(0) \ni D \cap \cup B. \)

Analogously \( \mathfrak{C} \neq \emptyset \Rightarrow C(0) = D \cap \cup C. \)

Let \( D \cap \cup B = B(0) \) and \( D \cap \cup C = C(0). \) Then \( B(0) \cap \cup C = D \cap \cup B \cap \cup C = \) 
\( \emptyset = C(0) \cap \cup B. \)
References


Author's address: 662 95 Brno, Janáčkovo nám. 2a (Přírodovědecká fakulta UJEP).