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TWO PROBLEMS CONCERNING INVERSE ANALYTIC FUNCTIONS

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One of the main problems is the validity of the identity $(\mathscr{F}_{-1})_{-1} = \mathscr{F}$ for a given analytic function $\mathscr{F}$ in an arbitrary region Ω. (If only analytic functions in the whole (extended) Gaussian plane $\mathbf{S}$ are admitted, there is, of course, no such problem; the identity holds for every non-constant analytic function in $\mathbf{S}$ [1].) One of the main practical problems is the question whether $\mathscr{F}_{-1}$ admits unrestricted continuation (e.g. in its "natural region"[1]). The well-applicable Theorem 3 answers these questions.

1. Denote by $\mathbf{S}$ and $\mathbf{E}$ the closed (extended) and open Gaussian plane, respectively. For each $z \in \mathbf{E}$, $\Delta \in (0, \infty\rangle$, let $U(z, \Delta) = \{z' \in \mathbf{E};\ |z' - z| < \Delta\}$ be the Δ-neighbourhood of z; further, let $U(\infty, \Delta) = \mathrm{Id}^{-1}(U(0, \Delta))$ (for each $\Delta \in ((0, \infty\rangle$; Id is the identical mapping, $\mathrm{Id}^{-1} = 1/\mathrm{Id}$).

An analytic element is every pair $[F, a]$ where $a \in \mathbf{S}$ and where F is meromorphic at the point a; two such pairs $[F, a]$, $[G, b]$ are considered equal, iff $a = b$ and $F \equiv G$ in a neighbourhood $U(a)$ of a [1]. $\mathscr{E} = [F, a]$ being an (analytic) element put

$$(1) \qquad \mathbf{s}(\mathscr{E}) = a\,, \quad \mathbf{h}(\mathscr{E}) = F(a)\,.$$

For every non-empty region $\Omega \subset \mathbf{S}$ denote by $\mathfrak{E}(\Omega)$ the set of all elements $\mathscr{E}$ with $\mathbf{s}(\mathscr{E}) \in \Omega$. For every $\mathscr{E}_0 \in \mathfrak{E}(\Omega)$ with $\mathbf{s}(\mathscr{E}_0) = a$ and for every $\Delta \in (0, \infty\rangle$ with $U(a, \Delta) \subset \subset \Omega$ let

$$(2) \quad O(\mathscr{E}_0, \Delta) = \{\mathscr{E} \in \mathfrak{E}(\Omega);\ \mathbf{s}(\mathscr{E}) \in U(a, \Delta), \mathscr{E} \text{ is a direct continuation}^{1)} \text{ of } \mathscr{E}_0\}.$$

These neighbourhoods define a topology in $\mathfrak{E}(\Omega)$; the topological space $\mathfrak{E}(\Omega)$ is locally connected. Denote by $\mathfrak{A}(\Omega)$ the system of all components of $\mathfrak{E}(\Omega)$. Then, as is well known[1],

(3) $\mathscr{F} \in \mathfrak{A}(\Omega)$ means $\mathscr{F}$ is an analytic function in Ω,

(4) each $\mathscr{F} \in \mathfrak{A}(\Omega)$ is an arcwise connected open subspace of $\mathfrak{E}(\Omega)$ with a countable basis[2].

[1]) See *Saks-Zygmund*: Analytic Functions, 1952.

[2]) The famous Poincaré-Volterra Theorem.

We easily see that

(5) the mappings $\mathbf{s} : \mathfrak{E}(\mathbf{S}) \to \mathbf{S}$, $\mathbf{h} : \mathfrak{E}(\mathbf{S}) \to \mathbf{S}$ are continuous,

(6′) $\mathbf{s}(\mathscr{F})$ [3]) is a region for each $\mathscr{F} \in \mathfrak{A}(\Omega)$,

(6″) $\mathbf{h}(\mathscr{F})$ [4]) is a region for each non-constant $\mathscr{F} \in \mathfrak{A}(\Omega)$.

2. F being a meromorphic function in a (non-empty) region $\Omega \subset \mathbf{S}$, the set of all elements $\mathscr{E}_z = [F, z]$, $z \in \Omega$, is an analytic function $\mathscr{F}_F \in \mathfrak{A}(\Omega)$. $\mathscr{F} \in \mathfrak{A}(\Omega)$ being a single-valued analytic function, the function $F_{\mathscr{F}} : \mathbf{s}(\mathscr{F}) \to \mathbf{S}$ defined by the condition

(7) $F_{\mathscr{F}}(z) = \mathbf{h}(\mathscr{E}_z)$ where $\mathscr{E}_z \in \mathscr{F}$ is the (only) element with $\mathbf{s}(\mathscr{E}_z) = z$

is meromorphic in $\mathbf{s}(\mathscr{F})$.

We *identify* the functions F, $\mathscr{F}_F$, and $\mathscr{F}$, $F_{\mathscr{F}}$, respectively.

Remark. Let $\mathscr{F} \in \mathfrak{A}(\Omega)$ and let Ω^* be a region containing Ω. It may occur that the extension $\mathscr{F}^*$ of $\mathscr{F}$ onto Ω^* (i.e. the analytic function in Ω^* containing all elements $\mathscr{E} \in \mathscr{F}$) contains exactly the same elements as $\mathscr{F}$. Then, of course, $\mathscr{F}^* = \mathscr{F}$; there is *no difference* between $\mathscr{F}$ and $\mathscr{F}^*$.

As a consequence, sin, e.g., is a (single-valued) analytic function both in $\mathbf{E}$ and $\mathbf{S}$; logarithm is an analytic function in $\mathbf{S}$, $\mathbf{E}$, $\mathbf{S} - \{0\}$, and $\mathbf{E} - \{0\}$.

3. Denote by $\mathfrak{E}_{\text{inv}}$ the set of all invertible[1]) analytic elements. By well known theorems, for each non-constant function $\mathscr{F} \in \mathfrak{A}(\Omega)$,

(8) the set $\mathscr{F} - \mathfrak{E}_{\text{inv}}$ is isolated in $\mathscr{F}$,

(9) the set $\mathscr{F} \cap \mathfrak{E}_{\text{inv}}$ is a region.

Hence,

(10) for each two elements $\mathscr{E}, \mathscr{E}^* \in \mathscr{F} \cap \mathfrak{E}_{\text{inv}}$ there is a curve φ in $\mathscr{F} \cap \mathfrak{E}_{\text{inv}}$ connecting $\mathscr{E}$ with $\mathscr{E}^*$.

For each $\mathscr{E} \in \mathfrak{E}_{\text{inv}}$, denote by $\mathscr{E}_{-1}$ the inverse element of $\mathscr{E}$. As is easily seen, the function $\chi : \mathfrak{E}_{\text{inv}} \to \mathfrak{E}_{\text{inv}}$ defined by $\chi(\mathscr{E}) = \mathscr{E}_{-1}$ is continuous, hence (in virtue of the identity $\chi_{-1} = \chi$) a homeomorphism. Thus,

(11) for each curve φ in $\mathfrak{E}_{\text{inv}}$, $\chi \circ \varphi$ also is a curve.

Definition. Let $\mathscr{F} \in \mathfrak{A}(\Omega)$ be a non-constant analytic function and let Ω^* be any region containing $\mathbf{h}(\mathscr{F} \cap \mathfrak{E}_{\text{inv}})$. An analytic function $\mathscr{F}^* \in \mathfrak{A}(\Omega^*)$ is called *the inverse of $\mathscr{F}$ in Ω^**, iff $\mathscr{F}^*$ contains at least one element of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{\text{inv}}$.

[3]) $\mathbf{s}(\mathscr{F})$ is the "natural region" (or "definition domain") of $\mathscr{F}$.

[4]) $\mathbf{h}(\mathscr{F})$ is the range (of values) of $\mathscr{F}$.

By (10) and (11),

(12) *for each region* $\Omega^* \supset h(\mathscr{F} \cap \mathfrak{E}_{inv})$ *there is one and only one inverse analytic function* $\mathscr{F}^*$ *of* $\mathscr{F}$ *in* Ω^*; *it contains all elements of the form* $\mathscr{E}_{-1}$ *where* $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{inv}$.

Definition. Let $\mathscr{F}^* \in \mathfrak{A}(\Omega^*)$ be the inverse of $\mathscr{F} \in \mathfrak{A}(\Omega)$ in Ω^* and $\mathscr{F}$ the inverse of $\mathscr{F}^*$ in Ω. Then we say that the functions $\mathscr{F}, \mathscr{F}^*$ are *mutually inverse* and write $\mathscr{F}^* = \mathscr{F}_{-1}$.[5])

Example 1. If $\mathscr{F}^* \in \mathfrak{A}(\mathbf{S})$ is the inverse of $\mathscr{F} \in \mathfrak{A}(\mathbf{S})$, then $\mathscr{F}, \mathscr{F}^*$ are mutually inverse. (Hence, exp and log, Id^n and the n-th root are mutually inverse.)

Example 2. The (only) branch[1]) $\mathscr{F}^*$ of logarithm in $P(0, 1) = U(0, 1) - \{0\}$ is the inverse of $\mathscr{F} = \exp \mid \{z \in \mathbf{E};\ \operatorname{Re} z < 0,\ |\operatorname{Im} z| < 2\pi\}$, but $\mathscr{F}, \mathscr{F}^*$ *are not* mutually inverse. Denoting by $\mathscr{F}^{**}$ the inverse of $\mathscr{F}^*$ in the half-plane $h(\mathscr{F}^* \cap \mathfrak{E}_{inv}) = h(\mathscr{F}^*) = \{z \in \mathbf{E};\ \operatorname{Re} z < 0\}$ we have $\mathscr{F}^{**} = \exp \mid \{z \in \mathbf{E};\ \operatorname{Re} z < 0\} \neq \mathscr{F}$. $\mathscr{F}^*$ and $\mathscr{F}^{**}$ are mutually inverse.

4. Definition. We say that $\mathscr{F} \in \mathfrak{A}(\Omega)$ is *univalent*[1]), iff $h \mid \mathscr{F}$ is one-one.

As is easily seen,

(13) *the univalence of* $\mathscr{F} \in \mathfrak{A}(\Omega)$ *implies* $\mathscr{F} \subset \mathfrak{E}_{inv}$.

Theorem 1. *If* $\mathscr{F} \in \mathfrak{A}(\Omega)$ *is univalent, then the inverse function* $\mathscr{F}^*$ *of* $\mathscr{F}$ *in* $h(\mathscr{F})$ *is meromorphic and contains exactly all elements of the form* $\mathscr{E}_{-1}$ *where* $\mathscr{E} \in \mathscr{F}$; *further,* $\mathscr{F}, \mathscr{F}^*$ *are mutually inverse.*

If a non-constant function $\mathscr{F} \in \mathfrak{A}(\Omega)$ *is not univalent, then the inverse function* $\mathscr{F}^*$ *of* $\mathscr{F}$ *in* $h(\mathscr{F})$ *is not single-valued.*

Proof. 1. Let $\mathscr{F} \in \mathfrak{A}(\Omega)$ be univalent. Then for each $w \in h(\mathscr{F})$ there is exactly one element $\mathscr{E}^w \in \mathscr{F}$ with $h(\mathscr{E}^w) = w$. Denoting

$$H(w) = s(\mathscr{E}^w) \quad \text{for each} \quad w \in h(\mathscr{F}), \tag{14}$$

we define a mapping H of $h(\mathscr{F})$ onto $s(\mathscr{F})$; let us see that H is meromorphic in $h(\mathscr{F})$.

Choose $w \in h(\mathscr{F})$ arbitrarily and denote $z = s(\mathscr{E}^w)$; then there is a $\Delta > 0$ such that $U = U(z, \Delta) \subset \Omega$, and a conformal mapping $F : U \to \mathbf{S}$ such that $\mathscr{E}^w = [F, z]$. $F(U)$ is a region contained in $h(\mathscr{F})$ and containing $F(z) = h(\mathscr{E}^w) = w$. For each $w' \in F(U)$ there is a $z' \in U$ with $F(z) = w'$. As $[F, z] \in \mathscr{F}$, we have $[F, z'] = \mathscr{E}^{w'}$ and $H(w') = z' = F_{-1}(w')$. Therefore, $H = F_{-1}$ in $F(U)$ and $[H, w] = (\mathscr{E}^w)_{-1}$.
Hence, H is meromorphic at the (arbitrary) point $w \in h(\mathscr{F})$. Moreover, $\mathscr{F}^* = H$ is the inverse of $\mathscr{F}$ in $h(\mathscr{F})$ (containing, of course, only elements of the form $[H, w]$, $w \in h(\mathscr{F})$, hence, only elements of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F}$). We easily see that,

[5]) The condition being symmetrical we have $(\mathscr{F}_{-1})_{-1} = \mathscr{F}$, of course.

reversely, $\mathscr{F}$ is the inverse function of H. Thus, $\mathscr{F}$ and $\mathscr{F}^* = H$ are mutually inverse.

2. Now let us suppose that the non-constant function $\mathscr{F} \in \mathfrak{A}(\Omega)$ is not univalent. Then there are two distinct elements $\mathscr{E}_j \in \mathscr{F}$ $(j = 1, 2)$ with $\mathbf{h}(\mathscr{E}_1) = \mathbf{h}(\mathscr{E}_2)$. Investigating separately two situations: 1. $\mathbf{s}(\mathscr{E}_1) = \mathbf{s}(\mathscr{E}_2)$, 2. $\mathbf{s}(\mathscr{E}_1) \neq \mathbf{s}(\mathscr{E}_2)$, in each case we easily find (in any neighbourhood of $\mathscr{E}_j$) two distinct invertible elements $\mathscr{E}^j \in \mathscr{F}$ with $\mathbf{h}(\mathscr{E}^1) = \mathbf{h}(\mathscr{E}^2)$. Then $\mathscr{E}_j^* = (\mathscr{E}^j)_{-1} \in \mathscr{F}^*$ for $j = 1, 2$, $\mathscr{E}_1^* \neq \mathscr{E}_2^*$, and $\mathbf{s}(\mathscr{E}_1^*) = \mathbf{h}(\mathscr{E}^1) = \mathbf{h}(\mathscr{E}^2) = \mathbf{s}(\mathscr{E}_2^*)$; therefore, $\mathscr{F}^*$ is not single-valued.

Theorem 2. *Let $\mathscr{F} \in \mathfrak{A}(\Omega)$, $\mathscr{F}^* \in \mathfrak{A}(\Omega^*)$ be mutually inverse functions, $\mathscr{F}$ being single-valued (i.e., meromorphic in $\mathbf{s}(\mathscr{F})$); put*

$$(15) \qquad \Omega_1 = \mathbf{s}(\mathscr{F} \cap \mathfrak{E}_{\mathrm{inv}}) .$$

Then $\mathscr{F}^*$ is univalent, contains exactly all elements of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{\mathrm{inv}}$, and

$$(16) \qquad \mathbf{s}(\mathscr{F}^*) = \mathscr{F}(\Omega_1) , \quad \mathbf{h}(\mathscr{F}^*) = \Omega_1 .$$

Proof. $\mathscr{F}^*$ being the inverse of $\mathscr{F}$, it contains all elements of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{\mathrm{inv}}$. For each $\mathscr{E}^* \in \mathscr{F}^* \cap \mathfrak{E}_{\mathrm{inv}}$, the element $\mathscr{E} = (\mathscr{E}^*)_{-1}$ belongs to $\mathscr{F}$ (as $\mathscr{F}$ is the inverse of $\mathscr{F}^*$), and $\mathscr{E}^* = \mathscr{E}_{-1}$. In order to see that $\mathscr{F}^*$ contains exactly all elements of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{\mathrm{inv}}$, it remains prove that $\mathscr{F}^*$ contains no non-invertible elements.

Suppose, on the contrary, that there is an $\mathscr{E}_0 \in \mathscr{F}^* - \mathfrak{E}_{\mathrm{inv}}$. The set $\mathscr{F}^* - \mathfrak{E}_{\mathrm{inv}}$ being isolated in $\mathscr{F}^*$, there is a $\delta > 0$ such that each element $\mathscr{E} \in O(\mathscr{E}_0, \delta)$, $\mathscr{E} \neq \mathscr{E}_0$, is invertible. Writting $\mathscr{E}_0 = [F, a]$, F is not conformal at the point a and, therefore, there are two distinct points $z_1, z_2 \in U(a, \delta)$, $z_1 \neq a \neq z_2$, such that $F(z_1) = F(z_2)$. Then $\mathscr{E}_j = [F, z_j]$, $j = 1, 2$, are distinct invertible elements of $\mathscr{F}^*$, and $\mathscr{E}^j = (\mathscr{E}_j)_{-1} \in \mathscr{F}$ are two distinct elements with $\mathbf{s}(\mathscr{E}^1) = \mathbf{s}(\mathscr{E}^2)$. Hence, $\mathscr{F}$ is not single-valued — a contradiction.

$\mathscr{F}^*$ containing exactly all elements of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{\mathrm{inv}}$, the identities (16) hold.

Suppose now that $\mathscr{F}^*$ is not univalent; then, by Theorem 1, the inverse function $\mathscr{F}^{**}$ of $\mathscr{F}^*$ in $\mathbf{h}(\mathscr{F}^*) = \Omega_1$ is not single-valued. As evidently $\mathscr{F}^{**} = \mathscr{F} \mid \Omega_1$, we obtain a contradiction.

5. **Theorem 3.** *Let $\mathscr{F} \in \mathfrak{A}(\Omega)$ be a single-valued function and let $\mathscr{F}^* \in \mathfrak{A}(\Omega^*)$ be the inverse function of $\mathscr{F}$ in Ω^*. Ω_1 being the set (15), let $\Omega_1^* \subset \mathscr{F}(\Omega_1)$ be a non-empty region. Further, suppose the following condition holds:*

(17) *For each $w \in \Omega_1^*$, there is a region $G \subset \Omega_1^*$ containing w, and further, there are disjoint regions H_z, $z \in \mathscr{F}_{-1}(w)$, such that $\mathscr{F}_{-1}(G)$ [6]) is a subset of the*

[6]) $\mathscr{F}_{-1}(w)$, $\mathscr{F}_{-1}(G)$ are the inverse-images of the point w and the set G, respectively, under (the meromorphic function) $\mathscr{F}$.

union of all regions H_z, that G is a subset of the intersection of all regions $\mathscr{F}(H_z)$, $z \in \mathscr{F}_{-1}(w)$, and moreover, $z \in H_z$ and $\mathscr{F} \mid H_z$ is one-one for each $z \in \mathscr{F}_{-1}(w)$.

Then the following assertions hold:

1. *Each branch*[1]) *$\mathscr{F}_1^*$ of $\mathscr{F}^*$ in Ω_1^* containing some element of the form $\mathscr{E}_{-1}$ where $\mathscr{E} \in \mathscr{F} \cap \mathfrak{E}_{\text{inv}}$, $\mathbf{s}(\mathscr{E}) \in \Omega_1$, admits unrestricted continuation in Ω_1^* and contains only elements of the above form.*

2. *If $\Omega^* = \mathbf{h}(\mathscr{F})$, if $\mathscr{F} \subset \mathfrak{E}_{\text{inv}}$, and if the condition (17) holds for $\Omega_1^* = \Omega^*$, then $\mathscr{F}, \mathscr{F}^*$ are mutually inverse.*

3. *If $\mathscr{F}, \mathscr{F}^*$ are mutually inverse functions, then $\mathscr{F}^*$ admits unrestricted continuation in Ω_1^*.*

4. *The condition (17) holds, if*

(18) *there is a set $\mathfrak{S}$ of regions the union of which is Ω_1^* such that for each $G \in \mathfrak{S}$ the function $\mathscr{F}$ is one-one on each component of the set $\mathscr{F}_{-1}(G)$ and maps it onto G.*

Proof. 1. Let the assumptions of part 1 of the theorem hold and let $\mathscr{F}_1^*$ be a branch of $\mathscr{F}^*$ in Ω_1^* containing an element $\mathscr{E}^* = \mathscr{E}_{-1}$ where $\mathscr{E} = [F, z] \in \mathscr{F} \cap \mathfrak{E}_{\text{inv}}$, $z \in \Omega_1$. It is sufficient to prove that the element $\mathscr{E}^* = [F^*, a]$ admits a continuation along every curve[1]) $\varphi : \langle \alpha, \beta \rangle \to \Omega_1^*$ with $\varphi(\alpha) = a$, and this continuation is an element of the same form as $\mathscr{E}^*$.

Denote by M the set of all $\tau \in (\alpha, \beta\rangle$ for which the element $\mathscr{E}^*$ admits a continuation along the restricted curve $\varphi \mid \langle \alpha, \tau \rangle$, the respective chain[1]) containing only inverse elements of invertible elements of $\mathscr{F}$.

If $\tau > \alpha$ is sufficiently close to α, the elements of the form $[F^*, \varphi(t)]$, $\alpha \leqq t \leqq \tau$, form such a chain. Hence $M \neq \emptyset$ and $c = \sup M \in (\alpha, \beta\rangle$; put $N = F_{-1}(\varphi(c))$.

As $\varphi(c) \in \Omega_1^*$, by (17) there is a region $G \subset \Omega_1^*$ containing $\varphi(c)$, and a system of regions H_z, $z \in N$, such that $z \in H_z$ and $F \mid H_z$ is one-one for each $z \in N$, and

$$G \subset \bigcap_{z \in N} \mathscr{F}(H_z), \quad \mathscr{F}_{-1}(G) \subset \bigcup_{z \in N} H_z . \tag{19}$$

Choose $\gamma \in M \cap (\alpha, c)$ so that $\varphi(\langle \gamma, c \rangle) \subset G$, and let $\{\mathscr{E}_t^*\}_{\alpha \leqq t \leqq \gamma}$ be the chain of elements along $\varphi \mid \langle \alpha, \gamma \rangle$ starting with $\mathscr{E}_\alpha^* = \mathscr{E}^*$, each element of the chain being the inverse of an (invertible) element of $\mathscr{F}$. Let $\mathscr{E}_\gamma^* = [\Psi, \varphi(\gamma)]$, where $\Psi : U(\varphi(\gamma)) \to \mathbf{S}$ is a conformal mapping, $U(\varphi(\gamma)) \subset G$. Then Ψ is the inverse-function of $\mathscr{F} \mid \Psi(U(\varphi(\gamma)))$, and the connected set $\Psi(U(\varphi(\gamma)))$ is contained in $\mathscr{F}_{-1}(G)$. By the second inclusion in (19) and by the disjointness of the regions H_z, $\Psi(U(\varphi(\gamma))) \subset H_z$ for some $z \in N$. $\mathscr{F} \mid H_z$ is one-one and equal to Ψ_{-1} in $\Psi(U(\varphi(\gamma)))$. It follows that the meromorphic function $(\mathscr{F} \mid H_z)_{-1}$ is an extension of Ψ onto the region $\mathscr{F}(H_z)$ which contains, by the first inclusion in (19), the region G.

$\{[(\mathscr{F} \mid H_z)_{-1}, \varphi(t)]\}_{\gamma \leqq t \leqq c}$ is a chain along the curve $\varphi \mid \langle \gamma, c \rangle$ starting with $\mathscr{E}_\gamma^*$ (which is a continuation of $\mathscr{E}^*$ along the curve $\varphi \mid \langle \alpha, \gamma \rangle$). Therefore, $c \in M$; supposing

$c < \beta$ we could (by means of elements of the form $[(\mathscr{F} \mid H_z)_{-1}, \varphi(t)]$) continuate further, beyond c, which would be a contradiction with the definition of c. Hence, $c = \beta$; $\mathscr{E}^*$ admits a continuation along φ and all elements of the respective chain have the required form.

2. By the assumption of part 2 of the theorem, $\mathscr{F}^*$ has in $\mathbf{h}(\mathscr{F}) = \Omega_1^*$ only one branch (identical with $\mathscr{F}^*$) and contains all elements of the form $\mathscr{E}_{-1}$ where $\mathscr{E} = [F, z]$, $z \in \Omega_1 = \mathbf{s}(\mathscr{F})$. By part 1 of the theorem, it does not contain any other elements. This implies that $\mathbf{s}(\mathscr{F}^*) = \mathbf{h}(\mathscr{F})$, $\mathbf{h}(\mathscr{F}^*) = \mathbf{s}(\mathscr{F})$; therefore the function $\mathscr{F}, \mathscr{F}^*$ are mutually inverse.

3. If $\mathscr{F}, \mathscr{F}^*$ are mutually inverse functions, each brach $\mathscr{F}_1^*$ of $\mathscr{F}^*$ in Ω_1^* contains inverse elements of invertible elements of $\mathscr{F}$, for it contains invertible elements $\mathscr{E}^*$ and for such elements $(\mathscr{E}^*)_{-1} \in \mathscr{F}$. By part 1 of the theorem, each branch $\mathscr{F}_1^*$ admits unrestricted continuation in Ω_1^*; the same holds for $\mathscr{F}^*$.

4. Now suppose the validity of (18) and let $w \in \Omega_1^*$; let $G \in \mathfrak{S}$ be a region containing w. By assumption, $\mathscr{F}(H) = G$ for any component H of $\mathscr{F}_{-1}(G)$. Therefore, H contains a point z with $\mathscr{F}(z) = w$, i.e., a point $z \in \mathscr{F}_{-1}(w)$. As $\mathscr{F} \mid H$ is one-one by assumption, H contains only one such point. This implies that there is a one-one correspondence between the components of the set $\mathscr{F}_{-1}(G)$ and the points z from $\mathscr{F}_{-1}(w)$ such that each z is contained in the corresponding component of $\mathscr{F}_{-1}(G)$. Not only inclusions (19), where $N = \mathscr{F}_{-1}(w)$, but equalities hold.

Example 3. arcsin being the inverse analytic function of $\sin \in \mathfrak{A}(\mathbf{S})$ in $\mathbf{S}$, the functions sin and arcsin are mutually inverse. By Theorem 2, arcsin is univalent and contains exactly all elements $\mathscr{E}_{-1}$ where $\mathscr{E} = [\sin, z]$, $z \in \mathbf{E}$, $z \neq \frac{1}{2}(2k + 1)\pi$ (k being an integer). Hence, for instance,

(20) $$\mathbf{s}(\arcsin) = \mathbf{E} - \{-1, +1\}\,.$$

Denote

(21) $$G = \mathbf{E} - ((-\infty, -1\rangle \cup \langle 1, \infty)),\; G' = \mathbf{E} - (-\infty, 1\rangle,\; G'' = \mathbf{E} - \langle -1, \infty)$$

and

(22) $$H_n = \{z \in \mathbf{E};\; \tfrac{1}{2}(2n - 1)\pi < \operatorname{Re} z < \tfrac{1}{2}(2n + 1)\pi\}\,,$$

(22') $$H'_n = \{z \in \mathbf{E};\; \tfrac{1}{2}(2n - 1)\pi < \operatorname{Re} z < \tfrac{1}{2}(2n + 3)\pi,\; \operatorname{Im} z > 0\}\,,$$

(22'') $$H''_n = \{z \in \mathbf{E};\; \tfrac{1}{2}(2n - 1)\pi < \operatorname{Re} z < \tfrac{1}{2}(2n + 3)\pi,\; \operatorname{Im} z < 0\}\,,$$

where n is an integer. Then sin is one-one in each of the regions H_n, H'_n, H''_n and

(23) $$\sin_{-1}(G) = \bigcup_{n=-\infty}^{+\infty} H_n\,,\quad \sin_{-1}(G') = \bigcup_{n=-\infty}^{+\infty} (H'_{2n} \cup H''_{2n})\,,$$

$$\sin_{-1}(G'') = \bigcup_{n=-\infty}^{+\infty} (H'_{2n+1} \cup H''_{2n+1})\,,$$

(24) $$\sin(H_n) = G\,, \quad \sin(H'_{2n}) = \sin(H''_{2n}) = G'\,,$$
$$\sin(H'_{2n+1}) = \sin(H''_{2n+1}) = G''\,.$$

The system $\mathfrak{S} = \{G, G', G''\}$ satisfies all conditions of part 4 of Theorem 3 with $\Omega_1^* = \mathbf{s}(\arcsin)$; by part 3 of Theorem 3,

(25) arcsin *admits unrestricted continuation in* $\mathbf{s}(\arcsin)$.

Example 4. Let $\mathscr{F}^* \in \mathfrak{A}(\mathbf{S})$ be the inverse analytic function of a (non-constant) rational function $\mathscr{F}$; then $\mathscr{F}$, $\mathscr{F}^*$ are mutually inverse.

Denoting $\Omega_1 = \mathbf{s}(\mathscr{F} \cap \mathfrak{E}_{\text{inv}})$, $\Omega_2 = \mathbf{s}(\mathscr{F} - \mathfrak{E}_{\text{inv}})$, the set Ω_2 is finite. Let us prove that

(26) $\mathscr{F}^*$ *admits unrestricted continuation in* $\mathbf{S} - \mathscr{F}(\Omega_2)$.

(Corollary: If $\mathscr{F}(\Omega_1) \cap \mathscr{F}(\Omega_2) = \emptyset$, then $\mathscr{F}^*$ admits unrestricted continuation in $\mathbf{s}(\mathscr{F}^*)$. It may be proved that the condition is not only sufficient, but also necessary.)

Let $w \in \mathbf{S} - \mathscr{F}(\Omega_2)$ be an arbitrary point; let $\mathscr{F}_{-1}(w) = \{a_1, \ldots, a_p\}$, where a_j are distinct points. All points a_j belong to Ω_1; hence, there is a $\Delta > 0$ such that $U(a_j, \Delta)$ are disjoint neighbourhoods and that each restriction $\mathscr{F} \mid U(a_j, \Delta)$ is one-one. As we easily see,

(27) $$\mathscr{F}_{-1}(U(w, \delta)) \subset \bigcup_{j=1}^{p} U(a_j, \Delta)$$

for each sufficiently small $\delta > 0$.

Each of the open sets $\mathscr{F}(U(a_j, \Delta))$ contains the point $w = \mathscr{F}(a_j)$; hence there is a $\delta > 0$ such that (27) holds and moreover,

(28) $$U(w, \delta) \subset \bigcap_{j=1}^{p} \mathscr{F}(U(a_j, \Delta))\,.$$

Denoting $\Omega_1^* = \mathbf{S} - \mathscr{F}(\Omega_2)$ we see that the condition (17) holds; by part 3 of Theorem 3, (26) holds.

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