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ALGEBRAIC PROPERTIES OF HUSIMI TREES

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1. INTRODUCTION

By a graph we mean a finite undirected graph with no loops or multiple edges (i.e. a graph in the sense of the books [1] or [2]). If G is a graph, then $V(G)$ or $E(G)$ denotes the vertex set of G or the edge set of G, respectively. Following [1] we shall say that a graph G is a *Husimi tree* if it is connected and every block of G is a complete graph. (Note that the concept of a Husimi tree in our sense is different from that in the sense of [6]).

By a ternary algebra we mean an ordered pair (U, ω), where U is a nonempty set and ω is a mapping of $U \times U \times U$ into U. Let $A = (U, \omega)$ be a ternary algebra; then we shall write $V(A) = U$; if $r, s, t \in V(A)$, then instead of $\omega(r, s, t)$ we shall write rst or rst_A. We shall say that a ternary algebra A is an HT-algebra if $V(A)$ is finite and the following axioms hold (u, v, w, and x are arbitrary elements of $V(A)$):

I $\quad uvu = u$,

II $\quad uvw = wvu$,

IIIA $\quad uv(uvw) = uvw$,

IIIB $\quad u(uvw)\, w = uvw$,

IV $\quad (uvx)\, xw = u(vxw)\, x$,

V $\quad (uvw)\,(vuw)\, x \in \{uvw, vuw\}$,

VI $\quad |\{uxv, vxw, uxw\}| \leqq 2$.

(Clearly, every tree algebra in the sense of [4] is an HT-algebra.)

Let U be a finite nonempty set. In the present paper we shall show that there exists a one-to-one correspondence between the set of Husimi trees G with $V(G) = U$ and the set of HT-algebras A with $V(A) = U$.

We begin with a useful characterization of Husimi trees:

Proposition 1. *Let G be a graph. Then the following statements are equivalent:*

(1) *G is a Husimi tree;*

(2) *for any $u, v \in V(G)$, there exists exactly one induced $u - v$ path in G.*

Proof. (1) $\Rightarrow$ (2) is obvious.

non (1) $\Rightarrow$ non (2). Assume that G is not a Husimi tree. The case when G is disconnected is obvious. Let G be connected. Then there exists a noncomplete block F of G. Therefore, there exist distinct vertices u, v, and w of F such that $uw, vw \in E(F)$ and $uv \notin E(F)$. Consider a shortest cycle C containing u, v, and w. Clearly, $C - w$ is an induced $u - v$ path in G. This means that there exist at least two induced $u - v$ paths in G, which completes the proof.

Let G be a Husimi tree. For any $u, v \in V(G)$, we denote

$$[u. v]_G = \{x \in V(G);\ x \text{ belongs to the induced } u - v \text{ path in } G\} .$$

Moreover, for any $u, v \in V(G)$, we denote

$$[u, v]^*_G = \{x \in V(G);\ |[u, x]_G \cap [x, v]_G| = 1\} .$$

Proposition 2. *Let G be a Husimi tree. Then for any $u, v \in V(G)$,*

$$[u, v]^*_G = \{x \in V(G);\ x \textit{ belongs to a } u - v \textit{ path in } G\} .$$

Proof. If $x \in [u, v]^*_G$, then $[u, x]_G \cap [x, v]_G = 1$, and therefore, x belongs to a $u - v$ path in G.

Conversely, assume that x belongs to a $u - v$ path in G. Consider a path P which is a shortest path among the $u - v$ paths containing x. Denote

$$\text{P:}\quad u = w_1, \ldots, w_n = v .$$

Obviously, there exists k, $1 \leqq k \leqq n$, such that $w_k = x$. We denote by P_1 or P_2 the path $w_1, \ldots, w_k$ or the path $w_k, \ldots, w_n$, respectively. Both P_1 and P_2 are induced paths in G (or else P is not a shortest path among the $u - v$ paths containing x, which is a contradiction). This implies that

$$[u, x]_G = \{w_1, \ldots, w_k\} \quad \text{and} \quad [x, v]_G = \{w_k, \ldots, w_n\} .$$

Hence $x \in [u, v]^*_G$, which completes the proof.

Proposition 3. *Let G be a Husimi tree, and let $u, v, w \in V(G)$. Then*

$$|[u, v]_G \cap [v, w]_G \cap [u, w]^*_G| = 1 .$$

Proof. Denote $X = [u, v]_G \cap [v, w]_G \cap [u, w]_G$. If $u = w$, then $X = \{u\}$. Assume that $u \neq w$. If G is a complete graph then $[u, v]_G = \{u, v\}$, $[v, w]_G = \{v, w\}$ and $[u, w]^*_G = V(G)$; thus $X = \{v\}$.

We now assume that G is not a complete graph. This means that G contains at least two blocks and at least one cut-vertex. It is easy to see that there exists at most one cut-vertex s of G with the following property:
(*) each component of $G - s$ contains at most one of the vertices u, v, and w.
If there exists such a cut-vertex s, then $X = \{s\}$.

Assume that there exists no cut-vertex s with the property (*). Then there exists a block F of G such that each component of $G - V(F)$ contains at most one of the vertices u, v and w. It is easy to see that F is uniquely determined. If $v \in V(F)$, then $X = \{v\}$. Let $v \notin V(F)$. Then there exists exactly one cut-vertex t of G such that $t \in V(F)$ and the component of $G - t$ containing v contains neither u nor w. Then $X = \{t\}$, which completes the proof.

Let G be a Husimi tree. We denote by A_G the ternary algebra defined as follows: $V(A) = V(G)$ and

$$\{uvw_{A_G}\} = [u, v]_G \cap [v, w]_G \cap [u, w]^*_G, \quad \text{for any} \quad u, v, w \in V(G).$$

We shall say that A_G is the algebra of G.

Proposition 4. *The algebra of every Husimi tree is an HT-algebra.*

Proof. For any Husimi tree G_0, we denote by $b(G_0)$ the number of blocks of G_0.

Assume that G is a Husimi tree. We wish to prove that A_G is an HT-algebra. Let first $b(G) \leqq 1$. Then G is a complete graph. It is obvious that for any u, v, $w \in V(G)$,

$$uvw_{A_G} = v \quad \text{if} \quad u \neq w$$

and

$$uvw_{A_G} = u \quad \text{if} \quad u = w.$$

This implies that A_G fulfils Axioms I–VI.

Let now $b(G) \geqq 2$. Assume that for every Husimi tree G' with $b(G') < b(G)$ the proposition is proved. Since G is connected, there exists a cut-vertex t of G. Then there exist graphs F and H with the property that $V(F) \cap V(H) = \{t\}$, $V(F) \cup V(H) = V(G)$, $E(F) \cap E(H) = \emptyset$, and $E(F) \cup E(H) = E(G)$. Clearly, both F and H are Husimi trees and $\max(b(F), b(H)) < b(G)$. According to the induction assumption both A_F and A_H are HT-algebras.

Let $u, v, w \in V(G)$. We shall show that uvw_{A_G} can be determined by means of A_F or A_H. Without loss of generality we may assume that at most one of the vertices u, v, and w belongs to $V(H)$. Then we have

$$uvw_{A_G} = uvw_{A_F} \quad \text{if} \quad u, v, w \in V(F),$$
$$uvw_{A_G} = uvt_{A_F} \quad \text{if} \quad u, v \in V(F) \quad \text{and} \quad w \in V(H),$$

$$uvw_{A_G} = utw_{A_F} \quad \text{if} \quad u, w \in V(F) \quad \text{and} \quad v \in V(H), \quad \text{and}$$
$$uvw_{A_G} = tvw_{A_F} \quad \text{if} \quad u \in V(H) \quad \text{and} \quad v, w \in V(F).$$

Since both A_F and A_H are HT-algebras it is not difficult to see that A_G fulfils Axioms I–VI.

Remark 1. Let G be a connected graph. We denote by $\mathscr{H}(G)$ the graph with $V(\mathscr{H}(G)) = V(G)$ and such that vertices u and v are adjacent in $\mathscr{H}(G)$ if and only if $u \neq v$ and there exists a block F of G such that $u, v \in V(F)$. It is clear that for any connected graph G, (i) $\mathscr{H}(\mathscr{H}(G)) = \mathscr{H}(G)$, and (ii) $\mathscr{H}(G) = G$ if and only if G is a Husimi tree. The concept of the algebra of a Husimi tree can be generalized as follows: if G is a connected graph, then by the algebra of G we can mean the algebra of $\mathscr{H}(G)$. However, if G and G' are connected graphs and $\mathscr{H}(G) = \mathscr{H}(G')$, then the algebra of G is identical with that of G'.

Proposition 5. *The algebras of distinct Husimi trees are distinct.*

Proof. Let G and G' be distinct Husimi trees. If $V(G) \neq V(G')$, then $A_G \neq A_{G'}$. Assume that $V(G) = V(G')$. Since $G \neq G'$, without loss of generality we assume that $E(G) - E(G') \neq \emptyset$. Then there exist distinct $u, v \in V(G)$ such that $uv \in E(G)$ and $uv \notin E(G')$. Since G' is a Husimi tree, there exists a cut-vertex x of G' such that $u \neq x \neq v$ and each component of $G' - x$ contains at most one of the vertices u and v. This implies that $[x, v]_{G'} \cap [u, x]^*_{G'} = \{x\}$. Since $x \in [u, v]_{G'}$, we have that $uvx_{AG'} = x$. Since $uv \in E(G)$, we have that $[u, v]_G = \{u, v\}$, and therefore $uvx_{A_G} \in \{u, v\}$. This means that $uvx_{A_G} \neq uvx_{A_{G'}}$, and thus $A_G \neq A_{G'}$, which completes the proof.

3. HT-ALGEBRAS AND THEIR GRAPHS

In Propositions 6–9 we shall prove some properties of HT-algebras A which follow from Axioms I–IV and are independent of the fact that $V(A)$ is finite.

Proposition 6. *Let A be an HT-algebra, and let $u, v, w, x \in V(A)$. Then*

(a) $uuv = u$;
(b) $uvx = x \Rightarrow vux = x$;
(c) $vu(uvw) = uvw$;
(d) $u(uvw)\, v = uvw$;
(e) $uvw = vuw \Rightarrow uvw = uwv$;
(f) $uvx = uwx \Rightarrow vuw = vxw$.

Proof (application of Axioms I and II will not be mentioned explicitly).

(a) According to IV, $uuv = (uvu)\, uv = u(vuv)\, u = uvu = u$.

(b) If $uvx = x$, then it follows from IV that $vux = xuv = (uvx)\,uv = (xvu)\,uv = x(vuv)\,u = xvu = uvx = x$.

(c) follows from IIIA and (b).

(d) According to IV and (a), $u(uvw)\,v = (uuv)\,vw = uvw$.

(e) According to (d) and IV, $uwv = vwu = v(vwu)\,w = v(uwv)\,w = (vuw)\,wv$. If $vuw = uvw$, then according to (c) we have that $uwv = uvw$.

(f) Let $uvx = uwx$. According to (a), IIIA and IV, we have that $vxw = wxv = (wxv)\,(wxv)\,u = (wx(wxv))\,(wxv)\,u = w(x(wxv)\,u)\,(wxv) = w(u(vxw)\,x)\,(wxv)$. According to IV, $u(vxw)\,x = (uvx)\,xw$, and thus $vxw = w((uvx)\,xw)\,(wxv)$. Since $uvx = uwx$, it follows from (c), IV, IIIA, and IV that $vxw = w((uwx)\,xw)\,(wxv) = w(uwx)\,(wxv) = (vxw)\,(xwu)\,w = ((vxw)\,xw)\,wu = (vxw)\,wu = v(uwx)\,w$. Therefore, we have that $vxw = v(xwu)\,w$.

Analogously, we get that $vuw = v(uwx)\,w$. This implies that $vxw = vuw$, which completes the proof of the proposition.

Let A be an HT-algebra. For any $u, v \in V(A)$, we denote

$$[u, v]_A = \{x \in V(A);\ uvx = x\}$$

and

$$[u, v]_A^* = \{x \in V(A);\ uxv = x\}.$$

Instead of $[u, v]_A$ or $[u, v]_A^*$ we shall often write $[u, v]$ or $[u, v]^*$.

Proposition 7. *Let A be an HT-algebra, and let $u, v \in V(A)$. Then*

(a) $u \in [u, v]$;

(b) $[u, v] = [v, u]$;

(c) $|[u, u]| = 1$;

(d) $x \in [u, v] \Rightarrow [u, x] \subseteq [u, v]$;

(e) $[u, v] \subseteq [u, v]^*$.

Proof. (a), (b) and (c) easily follow from Proposition 6.

(d) Let $x \in [u, v]$ and $y \in [u, x]$. Since $[u, v] = [v, u]$, we have that $vux = x$ and $uxy = y$. According to IV, $vuy = yuv = (yxu)\,uv = y(xuv)\,u = yxu = y$. Hence $y \in [u, v]$.

(e) Let $x \in [u, v]$. Then $uvx = x$. According to Proposition 6(b), $vux = x$. Since $uvx = vux$, it follows from Proposition 6(e) that $uvx = uxv$. Hence $uxv = x$, and thus $x \in [u, v]^*$.

Proposition 8. *Let A be an HT-algebra, and let $u, v, w \in V(A)$. Then*

$$[u, v] \cap [v, w] \cap [u, w]^* = \{uvw\}.$$

Proof. Denote $X = [u, v] \cap [v, w] \cap [u, w]^*$. As follows from IIIA, IIIB and Proposition 6(c), $uv(uvw) = u(uvw)\,w = vw(uvw) = uvw$, and thus $uvw \in X$. Hence $X \neq \emptyset$.

Consider an arbitrary $x \in X$. Then $vux = vwx = uxw = x$. Since $vux = vwx$, it follows from Proposition 6(f) that $uvw = uxw$. Since $uxw = x$, we have that $x = uvw$, and thus $X = \{uvw\}$, which completes the proof.

Proposition 9. *Let A be an HT-algebra, and let $u, v \in V(A)$. Then*

$$[u, v]^* = \{x \in V(A);\ |[u, x] \cap [x, v]| = 1\}\,.$$

Proof. Denote $X = \{x \in V(A);\ |[u, x] \cap [x, v]| = 1\}$. Let first $x \in [u, v]^*$. Then $uxv = x$. Clearly, $x \in [u, x] \cap [x, v]$, and thus $[u, x] \cap [x, v] \neq \emptyset$. Consider an arbitrary $y \in [u, x] \cap [x, v]$. Then $uxy = y = xvy$. According to IV, we have that $y = uxy = ux(xvy) = (yvx)\,xu = y(vxu)\,x = x(uxv)\,y = xxy = x$. Hence $|[u, x] \cap [x, v]| = 1$, and thus $x \in X$.

Conversely, let $x \in X$. Then $|[u, x] \cap [x, v]| = 1$. As follows from Proposition 7, $[u, x] \cap [x, v] = \{x\}$. Denote $z = uxv$. According to Proposition 6, $uxz = xvz = z$, and thus $z \in [u, x] \cap [x, v]$. Since $[u, x] \cap [x, v] = \{x\}$, we have that $z = x$. Hence $uxv = x$, and thus $x \in [u, v]^*$, which completes the proof.

Remark 2. In the proofs of Propositions 6–9 only Axioms I–IV were used. Note that ternary algebras A fulfilling the property

$$uvw = vuw \quad \text{for any} \quad u, v, w \in V(A)$$

and Axioms I, II and IV are called normal graphic algebras in [5]. However, every normal graphic algebra fulfils also IIIA, IIIB and V.

Let A be an HT-algebra. We denote by G_A the graph defined as follows: $V(G_A) = V(A)$ and vertices u and v are adjacent in G_A if and only if $|[u, v]_A| = 2$.

In the proof of the next lemma we use only Axioms I–IV together with the fact that $V(A)$ is finite.

Lemma 1. *Let A be an HT-algebra. Then G_A is connected.*

Proof. (The idea of our proof is similar to that used in the proof of Proposition 7 in Mulder and Schrijver [3].) Consider arbitrary $u, v \in V(A)$. We wish to prove that there exists a $u - v$ path in G_A. Denote $n = |[u, v]_A|$. The case $n \leqq 2$ is obvious. Let $n \geqq 3$. Assume that for any $u', v' \in V(A)$ such that $|[u', v']_A| < n$ we have proved that there exists $u' - v'$ path in G_A. Since $n \geqq 3$, there exists $x \in [u, v]_A - \{u, v\}$. According to Propositions 7 and 9, $[u, x]_A \cup [x, v]_A \subseteq [u, v]_A$ and $[u, x]_A \cap [x, v]_A = \{x\}$. It follows from the induction assumption that there exist a $u - x$ path and an $x - v$ path in G_A. Hence, there exists a $u - v$ path in G_A, which completes the proof.

Lemma 2. *Let A be an HT-algebra, let $u, v \in V(A)$, and let P be an induced $u - v$ path in G_A. Then $[u, v]_A = V(P)$.*

Proof. Denote $n = |V(P)|$. The case $n \leqq 2$ is obvious. Let $n \geqq 3$. Assume that for any induced $u' - v'$ path P' in G_A with $|V(P')| < n$, we have proved that $V(P') = [u', v']_A$. Denote

$$P: \quad u = w_1, \ldots, w_{n-1}, \; w_n = v \, .$$

According to the induction assumption, $[w_1, w_{n-1}]_A = \{w_1, \ldots, w_{n-1}\}$. Obviously, $[w_{n-1}, w_n]_A = \{w_{n-1}, w_n\}$. Since $w_1 w_{n-1} w_n \in [w_1, w_{n-1}]_A \cap [w_n, w_{n-1}]_A$, we have that $w_1 w_{n-1} w_n = w_{n-1}$.

We wish to prove that $w_1 w_n w_{n-1} = w_{n-1}$. On the contrary, let $w_1 w_n w_{n-1} \neq w_{n-1}$. Since $[w_{n-1}, w_n]_A = \{w_{n-1}, w_n\}$, we have that $w_1 w_n w_{n-1} = w_n$. Since $w_1 w_{n-1} w_n \neq w_1 w_n w_{n-1}$, it follows from Proposition 6(e) that $w_{n-1} w_1 w_n \notin \{w_{n-1}, w_n\}$. Since $[w_{n-1}, w_1]_A = \{w_1, \ldots, w_{n-1}\}$, we have that there exists k, $1 \leqq k \leqq n - 2$, such that $w_{n-1} w_1 w_n = w_k$. It follows from V that for any $x \in V(A)$,

$$w_k w_n x = (w_{n-1} w_1 w_n)(w_1 w_n w_{n-1}) \, x \in \{w_k, w_n\} \, .$$

Therefore, $|[w_k, w_n]_A| = 2$. We have that $w_k w_n \in E(G_A)$, and thus P is not an induced path in G_A, which is a contradiction. This means that $w_1 w_n w_{n-1} = w_{n-1}$. Since $w_1 w_{n-1} w_n = w_{n-1}$, it follows from Proposition 6(e) that $w_{n-1} w_1 w_n = w_{n-1}$.

Since $w_{n-1} \in [w_1, w_n]_A$, it follows from Proposition 7(d) that $V(P) = \{w_1, \ldots, w_{n-1}\} \cup \{w_{n-1}, w_n\} \subseteq [u, v]_A$. We now wish to prove that $[u, v]_A \subseteq V(P)$. Let $x \in [u, v]_A$. Then $w_1 w_n x = w_n w_1 x = w_1 x w_n = x$. According to VI, we have that

$$|\{w_n w_1 w_{n-1}, w_n w_1 x, w_{n-1} w_1 x\}| \leqq 2 \quad \text{and} \quad \{w_1 w_n w_{n-1}, w_1 w_n x, w_{n-1} w_n x\}| \leqq 2 \, .$$

Assume that $x \notin V(P)$. Then $w_{n-1} w_1 x, w_{n-1} w_n x \in \{w_{n-1}, x\}$. If $w_{n-1} w_n x = w_{n-1} w_n x$, then it follows from Proposition 6(f) that $w_1 w_{n-1} w_n = w_1 x w_n$, and thus $x = w_{n-1}$, which is a contradiction. If $w_{n-1} w_1 x \neq w_{n-1} w_n x$, then either $w_{n-1} w_1 x = x$ or $w_{n-1} w_n x = x$, and thus $x \in V(P)$, which is a contradiction. Thus the proof is complete.

Proposition 10. *Let A be an HT-algebra. Then G_A is a Husimi tree and A is the algebra of G_A.*

Proof. According to Lemma 1, G_A is connected. It follows from Lemma 2 that for any $r, s \in V(A)$, there exists exactly one induced $r - s$ path in G_A and that $[r, s]_A$ is the vertex set of the induced $r - s$ path in G_A. According to Proposition 1, G_A is a Husimi tree. Moreover, for any $u, v \in V(A)$, $[u, v]_{G_A} = [u, v]_A$. According to Proposition 9, for any $u, v \in V(A)$, $[u, v]^*_{G_A} = [u, v]^*_A$. It follows from Proposition 8 that A is the algebra of G_A, which completes the proof.

4. THE MAIN RESULT

Let U be an arbitrary finite nonempty set. We denote by $\mathfrak{A}$ the set of HT-algebras A with $V(A) = U$, by $\mathfrak{G}$ the set of Husimi trees G with $V(G) = U$, and by $\mathfrak{G}_0$ the set of graphs G_0 with $V(G_0) = U$. Moreover, we denote by γ the mapping of $\mathfrak{A}$ into $\mathfrak{G}_0$ such that for every $A \in \mathfrak{A}$, $\gamma(A) = G_A$.

Theorem. *γ is a one-to-one mapping of $\mathfrak{A}$ onto $\mathfrak{G}$, and for every $G \in \mathfrak{G}$, $\gamma^{-1}(G)$ is the algebra of G.*

Proof follows from Propositions 4, 5 and 10.

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