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A NOTE ON TOLERANCE LATTICES

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Definitions. A *tolerance relation* is a reflexive and symmetric binary relation.

A *compatible tolerance* on an algebra $\mathfrak{A} = (A, F)$ is a tolerance relation on the support A, compatible with each operation $f \in F$.

An (a, b)-*maximal tolerance* on the algebra $\mathfrak{A}$ is such a compatible tolerance on $\mathfrak{A}$ that is maximal among all compatible tolerances on $\mathfrak{A}$ not containing the pair $[a, b]$.

A compatible tolerance T on the algebra $\mathfrak{A}$ is a *relatively maximal tolerance* if there exists a pair $[a, b] \in T^c = |\mathfrak{A}| \times |\mathfrak{A}| \setminus T$ such that T is an (a, b)-maximal tolerance.

Proposition 1. *Let T be a compatible tolerance on an algebra $\mathfrak{A}$. Let $[a, b] \in T^c$. Then there exists an (a, b)-maximal tolerance T_{ab} on $\mathfrak{A}$ such that $T \subseteq T_{ab}$.*

Proof. Denote by $\mathscr{T}_{ab}$ the set of all compatible tolerances on $\mathfrak{A}$ including T and not containing $[a, b]$. $\mathscr{T}_{ab} \neq \emptyset$ as $T \in \mathscr{T}_{ab}$. The union of each nested subset of $\mathscr{T}_{ab}$ is again an element of $\mathscr{T}_{ab}$. By Zorn lemma, $\mathscr{T}_{ab}$ has at least one maximal element T_{ab}.
Q.E.D.

A particular case of this assertion is the following result of Chajda and Zelinka:

Corollary. *Let $\mathfrak{A}$ be an algebra and let $a, b \in |\mathfrak{A}|$, $a \neq b$. Then there exists an (a, b)-maximal tolerance on $\mathfrak{A}$.* (Cf. [3], Thm. 4.)

Proposition 2. *Every compatible tolerance on an algebra $\mathfrak{A}$ is the intersection of a family of relatively maximal tolerances on $\mathfrak{A}$.*

Proof. Let T be a compatible tolerance on the algebra $\mathfrak{A}$. If $T^c = \emptyset$, $T = |\mathfrak{A}| \times |\mathfrak{A}| = \bigcap_{[a,b] \in T^c} T_{ab}$. Suppose $T^c \neq \emptyset$. Then $T \subseteq \bigcap_{[a,b] \in T^c} T_{ab}$, because $T \subseteq T_{ab}$ for all $[a, b] \in T^c$. Conversely, $\bigcap_{[a,b] \in T^c} T_{ab} \subseteq T$, because $[x, y] \notin T$ implies $[x, y] \notin T_{xy}$ and $[x, y] \in T^c$ and therefore $[x, y] \notin \bigcap_{[a,b] \in T^c} T_{ab}$. Hence $T = \bigcap_{[a,b] \in T^c} T_{ab}$.
Q.E.D.

Corollary. *For every algebra* $\mathfrak{A}$ *the following assertions are equivalent*:

(i) *every compatible tolerance on* $\mathfrak{A}$ *is a congruence*;

(ii) *every relatively maximal tolerance on* $\mathfrak{A}$ *is a congruence.*

Proof. (i) $\Rightarrow$ (ii): Clear.

(ii) $\Rightarrow$ (i): Every intersection of congruences is a congruence. Q.E.D.

Remark. Relatively maximal tolerances on $\mathfrak{A}$ are exactly all completely meet irreducible elements in the lattice of all compatible tolerances on $\mathfrak{A}$: As each completely meet irreducible element in the lattice of all compatible tolerances is the intersection of relatively maximal tolerances, it is itself a relatively maximal tolerance. Conversely, if an (a, b)-maximal tolerance T is the intersection of a family of compatible tolerances, at least one member of this family must not contain $[a, b]$, so it is identical with T.

Proposition 3. *For every algebra* $\mathfrak{A}$, *every subalgebra* $\mathfrak{B}$ *and every compatible tolerance* T *on* $\mathfrak{B}$ *the following assertion holds*: *If every relatively maximal tolerance on* $\mathfrak{B}$ *including* T *has an extension onto* $\mathfrak{A}$, T *also has an extension onto* $\mathfrak{A}$.

Proof. The intersection of extensions of T_{ab}, $[a, b] \in T^c = |\mathfrak{B}| \times |\mathfrak{B}| \setminus T$, $T \subseteq T_{ab}$ is an extension of T. Q.E.D.

Corollary. *For every algebra* $\mathfrak{A}$ *and every subalgebra* $\mathfrak{B}$ *the following assertions are equivalent*:

(i) *every compatible tolerance on* $\mathfrak{B}$ *has an extension onto* $\mathfrak{A}$;

(ii) *every relatively maximal tolerance on* $\mathfrak{B}$ *has an extension onto* $\mathfrak{A}$.

The structure of relatively maximal tolerances is known in the case of distributive lattices. They are a useful tool in studying compatible tolerances on distributive lattices. (Cf. [4].)

Definition. A (*dual*) *a-maximal ideal* in a lattice $\mathfrak{L}$ is a (dual) ideal in $\mathfrak{L}$ which is maximal among all (dual) ideals in $\mathfrak{L}$ not containing the element $a \in |\mathfrak{L}|$.

Proposition 4. *For a compatible tolerance* T *on a distributive lattice* $\mathfrak{L}$, *the following assertions are equivalent*:

(i) T *is an* (a, b)*-maximal tolerance*;

(ii) T *is a two-block tolerance* (*i.e.* T *has exactly two blocks*) *and* (a) *or* (a^d) *holds*:

  (a) *one of the blocks of* T *is an a-maximal ideal and the other one is a dual b-maximal ideal*;

  (a^d) *is the dual of* (a).

Proof. In [4], Lemma 2, it is shown that for every compatible tolerance T on the distributive lattice $\mathfrak{L}$ and for each pair of elements $[a, b] \in T^c$ there exists a compatible tolerance T^{ab} satisfying (ii), including T and not containing $[a, b]$.

(i) $\Rightarrow$ (ii): Because of maximality of T with respect to $[a, b]$, $T = T^{ab}$.

(ii) $\Rightarrow$ (i): Let T be formed by an a-maximal ideal I and a dual b-maximal ideal F. Let S be an (a, b)-maximal tolerance including T. Because of a-maximality of I and b-maximality of F, $S^{ab} = T$ and therefore $T = S$. Q.E.D.

Proposition 5. *For a lattice $\mathfrak{L}$, the following assertions are equivalent:*
(i) *$\mathfrak{L}$ is a distributive lattice;*
(ii) *all relatively maximal tolerances on $\mathfrak{L}$ are two-block tolerances.*

Proof. (i) $\Rightarrow$ (ii): By Proposition 4.

(ii) $\Rightarrow$ (i): Let $a, b, c \in |\mathfrak{L}|$ satisfy

$$a \wedge c = b \wedge c \quad \text{and} \quad a \vee c = b \vee c\,.$$

If T were a two-block (a, b)-maximal tolerance on $\mathfrak{L}$, T would be formed by an ideal I and a dual ideal F, with $I \cup F = |\mathfrak{L}|$. Two cases could arise: $a \notin I$ and $b \notin F$ or $a \notin F$ and $b \notin I$. In the first case, $a \wedge c = b \wedge c$ implies $c \notin F$ and $a \vee c = b \vee c$ implies $c \notin I$. The second case is the dual of the first one. Consequently, there exists no (a, b)-maximal tolerance on the lattice $\mathfrak{L}$. Hence $a = b$ and $\mathfrak{L}$ is a distributive lattice.

Q.E.D.

The congruence lattice of a lattice $\mathfrak{L}$ will be denoted by $CL(\mathfrak{L})$, the tolerance lattice of a lattice $\mathfrak{L}$ (i.e. the lattice of all compatible tolerances on $\mathfrak{L}$) will be denoted by $TL(\mathfrak{L})$.

If $\mathfrak{L}$ is a distributive lattice, then $CL(\mathfrak{L})$ is a Boolean lattice if and only if $\mathfrak{L}$ is a lattice of locally finite length. (J. Hashimoto; cf. [1], p. 80.)

If $\mathfrak{L}$ is a distributive lattice, then $\mathfrak{L}$ is a relatively complemented lattice if and only if every compatible tolerance on $\mathfrak{L}$ is a congruence ([2]).

Proposition 6. *If $\mathfrak{L}$ is a distributive lattice and if $TL(\mathfrak{L})$ is a Boolean lattice, then every compatible tolerance on $\mathfrak{L}$ is a congruence.*

Proof. Let T be an arbitrary relatively maximal tolerance on $\mathfrak{L}$, i.e. a two-block tolerance on $\mathfrak{L}$, let T^* be the complement of T in $TL(\mathfrak{L})$. Obviously, $T^* \neq \Delta$. Assume $I \cap F \neq \emptyset$, where I and F are the ideal and the dual ideal forming T. Take an arbitrary element $x \in I \cap F$. Let $[a, b] \in T^*$. Then $x \vee a \in F$, $x \vee b \in F$, $x \wedge a \in I$ and $x \wedge b \in I$. Hence $x \vee a = x \vee b$ and $x \wedge a = x \wedge b$, and the distributivity of $\mathfrak{L}$ yields $a = b$. Thus $T^* = \Delta$. This is a contradiction, consequently $I \cap F = \emptyset$ and T is a congruence. By Corollary of Proposition 2, every compatible tolerance on $\mathfrak{L}$ is a congruence. Q.E.D.

Corollary. *If $\mathfrak{L}$ is a distributive lattice, then $TL(\mathfrak{L})$ is a Boolean lattice if and only if $\mathfrak{L}$ is a relatively complemented lattice of locally finite length.*

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