Časopis pro pěstování matematiky

Ivan Kolář; Vladimír Lešovský

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*Časopis pro pěstování matematiky*, Vol. 107 (1982), No. 3, 253--256

Persistent URL: `http://dml.cz/dmlcz/118132`

# STRUCTURE EQUATIONS OF GENERALIZED CONNECTIONS

IVAN KOLÁŘ and VLADIMÍR LEŠOVSKÝ, Brno

(Received November 20, 1980)

Starting from some recent results by the first author, [2], [3], we deduce the structure equation of an arbitrary (generalized) connection on a fibered manifold with fiber parallelism. For a so-called homogeneous connection we obtain an interesting generalization of the classical structure equation of a principal connection. We also clarify that the homogeneity of the connection is an essential tool to deduce a kind of generalized Bianchi identity. – Our consideration is in the category $C^\infty$.

**1.** For any vector bundle $E \to X$, a linear base-preserving morphism $\varphi : \bigwedge^k TX \to E$ will be called an $E$-valued $k$-form. Given a linear connection $C$ on $E$, Koszul, [4], has defined the exterior differential $\mathrm{d}_C\varphi : \bigwedge^{k+1} TX \to E$. In some local coordinates $x^i$ on $X$ and some additional linear coordinates $z^p$ on $E$, if $\varphi^p$ are the components of $\varphi$ and $\Gamma^p_{qi}$ are Christoffel's symbols of $C$, then the components of $\mathrm{d}_C\varphi$ are

$$\mathrm{d}\varphi^p - \Gamma^p_{qi}\,\mathrm{d}x^i \wedge \varphi^q . \tag{1}$$

For $k = 1$, Koszul's formula reads

$$(\mathrm{d}_C\varphi)(\xi, \eta) = {}_C\nabla_\xi\, \varphi(\eta) - {}_C\nabla_\eta\, \varphi(\xi) - \varphi([\xi, \eta]) \tag{2}$$

for any vector fields $\xi$ and $\eta$ on $X$, provided ${}_C\nabla_\xi$ has the usual meaning of the absolute derivative.

Given a fibered manifold $p : Y \to X$, a linear base-preserving morphism $\varphi : \bigwedge^k TY \to E$ will be called an $E$-valued $k$-form on $Y$. Any linear connection $C$ on $E$ induces a linear connection $p^*C$ on the induced vector bundle $p^*E \to Y$, [1]. We define $\mathrm{d}_C\varphi := \mathrm{d}_{p^*C}\varphi$, where $\varphi$ on the right-hand side is interpreted as a map $\bigwedge^k TY \to p^*E$. Obviously, $\mathrm{d}_C\varphi$ can be regarded as an $E$-valued $(k + 1)$-form on $Y$. Formula (2) has now the form

$$(\mathrm{d}_C\varphi)(\xi, \eta) = {}_{p^*C}\nabla_\xi\, \varphi(\eta) - {}_{p^*C}\nabla_\eta\, \varphi(\xi) - \varphi([\xi, \eta]) \tag{3}$$

for any vector fields $\xi$ and $\eta$ on $Y$. An $E$-valued $k$-form $\varphi$ on $Y$ will be called horizontal if $\varphi(A_1, \ldots, A_k) = 0$ whenever at least one of the vectors $A_1, \ldots, A_k$ is vertical.

**2.** A fiber parallelism on a fibered manifold $p : Y \to X$ is a triple $(Y, E, Q)$, where $\pi : E \to X$ is a vector bundle over the same base $X$ and $Q : Y \oplus E \to VY$ is a morphism over $Y$ of the fiber product $Y \oplus E$ into the vertical tangent bundle $VY$ of $Y$ such that $Q(y) : E_{\pi(y)} \to V_y Y$ is a linear isomorphism for every $y \in Y$. Any vector $A \in E_x$ determines a vector field $QA$ on the fiber $Y_x$ and every section $\sigma : X \to E$ induces a vertical vector field $Q\sigma$ on $Y$. The structure function of $Q$ is a map $S_Q : Y \oplus \bigwedge^2 E \to E$ defined by

$$S_Q(y, A, B) = Q(y)^{-1} ([QA, QB]_y) . \tag{4}$$

A (generalized) connection on $Y$ means any section $\Gamma : Y \to J^1 Y$, where $J^1 Y$ denotes the first jet prolongation of $Y$, [5]. For every $y \in Y$, $\Gamma(y)$ is identified with a horizontal subspace in $T_y Y$ and any vector $A \in T_y Y$ is decomposed into $A = hA + vA$ with $hA \in \Gamma(y)$ and $vA \in V_y Y$. The connection form of $\Gamma$ is an $E$-valued 1-form $\omega$ on $Y$ determined by

$$\omega(A) = Q(y)^{-1} (vA) . \tag{5}$$

The curvature form of $\Gamma$ is a map $\Omega : Y \oplus \bigwedge^2 TX \to E$ defined by $\Omega(y, \xi_x, \eta_x) = -\omega([\Gamma\xi, \Gamma\eta]_y)$, $x = py$, for any vector fields $\xi$ and $\eta$ on $X$, provided $\Gamma\xi$ means the $\Gamma$-lift of $\xi$. Obviously, $\Omega$ can be regarded as a horizontal $E$-valued 2-form on $Y$. On the other hand, $d_C\omega$ is also an $E$-valued 2-form on $Y$.

**3.** We have to recall the concept of the deviation form $\delta(\Gamma, C, Q)$, [3]. Connections $\Gamma$ and $C$ determine the product connection $\Gamma \oplus C$ on $Y \oplus E$, which is transformed by $Q$ into a connection $Q(\Gamma \oplus C)$ on $VY$. On the other hand, $\Gamma$ is canonically prolonged into a connection $V\Gamma$ on $VY$, [2]. Under standard identifications, the difference $Q(\Gamma \oplus C) - V\Gamma$ can be interpreted as a map $\delta(\Gamma, C, Q) : Y \oplus E \oplus TX \to E$ linear in both $E$ and $TX$. Dualizing with respect to $E$, we can regard $\delta(\Gamma, C, Q)$ as a horizontal $E \otimes E^*$-valued 1-form on $Y$.

**Lemma 1.** *Given $A \in E_x$ and $B \in T_x X$, $x \in X$, let $\sigma$ be a section of $E$ with $j^1_x\sigma = C(A)$ and $\xi$ a vector field on $X$ with $\xi_x = B$. Then*

$$\delta(\Gamma, C, Q)(y, A, B) = \omega([\Gamma\xi, Q\sigma]_y) . \tag{6}$$

Proof consists in direct evaluation in local coordinates.

**4.** As usual, the symbol $\barwedge$ will denote the tensor contraction combined with alternation. Hence $\omega \barwedge \delta(\Gamma, C, Q)$ is an $E$-valued 2-form on $Y$. Analogously, the composition $S_Q(\omega, \omega)$ of the structure function of $Q$ and the connection form of $\Gamma$ can be regarded as an $E$-valued 2-form on $Y$.

**Theorem 1.** (*Structure equation.*) *We have*

$$d_C\omega = -S_Q(\omega, \omega) + \omega \barwedge \delta(\Gamma, C, Q) + \Omega . \tag{7}$$

Proof. By bilinearity, it is sufficient to discuss the value $(d_C\omega)(A, B)$ in the following three cases.

(i) Both $A$ and $B$ are vertical, so that the second and third terms on the right-hand side of (7) vanish. Let $A = (Q\sigma)_y$, $B = (Q\varrho)_y$ for some sections $\sigma$ and $\varrho$ of $E$. Then a simple calculation shows that the both absolute derivatives in (3) vanish. Hence $(d_C\omega)(A, B) = -\omega([Q\sigma, Q\varrho]_y)$, which is the required value of $S_Q(\omega, \omega)$.

(ii) Both $A$ and $B$ are horizontal, so that the first and second terms on the right-hand side of (7) vanish. Let $A = (\Gamma\xi)_y$ and $B = (\Gamma\eta)_y$ for some vector fields $\xi$ and $\eta$ on $X$. Then $\omega(\Gamma\xi) = \omega(\Gamma\eta) = 0$ and (3) implies $(d_C\omega)(A, B) = \Omega(A, B)$ by the definition of $\Omega$.

(iii) $A$ is vertical and $B$ is horizontal, so that the first and third terms on the right-hand side of (7) vanish. Let $A = (Q\sigma)_y$ for a section $\sigma$ of $E$ satisfying $j^1_x\sigma = C(A)$ and $B = (\Gamma\xi)_y$. In this case, one finds the following coordinate expression for ${}_{p^*C}\nabla_{\Gamma\xi}\, \omega(Q\sigma)$:

(8)
$$\frac{\partial\sigma^p}{\partial x^i}\xi^i - \Gamma^p_{qi}\sigma^q\xi^i\,,$$

where $\sigma^p(x)$ or $\xi^i(x)$ is the coordinate expression of $\sigma$ or $\xi$, respectively. But (8) vanishes at $x = py$ by the assumption $j^1_x\sigma = C(A)$. The second absolute derivative in (3) vanishes trivially, so that we have $(d_C\omega)(A, B) = -\omega([Q\sigma, \Gamma\xi]_y) = \delta(\Gamma, C, Q)(y, A, B)$ by Lemma 1, QED.

A connection $\Gamma$ on $Y$ is called homogeneous, [3], if there exists a linear connection $C$ on $E$ satisfying $\delta(\Gamma, C, Q) = 0$. In this case, $C$ is uniquely determined and is said to be associated with $\Gamma$. The structure equation of a homogeneous connection is

(9)
$$d_C\omega = -S_Q(\omega, \omega) + \Omega\,,$$

where $C$ is the associated connection. On every principal fiber bundle $P(X, G)$, there is a canonical fiber parallelism $N$ given by the classical fundamental vector fields on $P$, the corresponding vector bundle is $X \times \mathfrak{g}$ (= the Lie algebra of $G$). By Lemma 1, a (generalized) connection $\Gamma$ on $P$ is principal (i.e. right-invariant) iff $\delta(\Gamma, O, N) = 0$, where $O$ means the zero connection on the product bundle $X \times \mathfrak{g}$. The structure function of $N$ coincides with the bracket in $\mathfrak{g}$ and $d_O\omega$ is the classical exterior differential of a $\mathfrak{g}$-valued form, so that (9) is reduced to the classical structure equation of a principal connection.

**5.** Given $\Gamma$ and $C$ as above, the absolute exterior differential of an $E$-valued $k$-form $\varphi$ on $Y$ is defined by

$$D_C\varphi(A_1, \ldots, A_{k+1}) = d_C\varphi(hA_1, \ldots, hA_{k+1})\,.$$

**Lemma 2.** *For any $C$, we have*

(10)
$$D_C(d_C\omega) = 0\,.$$

Proof. Using (1), we find the following coordinate expression of $d_C(d_C\omega)$:

$$-d\Gamma^p_{qi} \wedge dx^i \wedge \omega^q + \Gamma^p_{qi}\Gamma^q_{rj}\, dx^i \wedge dx^j \wedge \omega^r, \tag{11}$$

which proves our assertion.

Quite similarly one deduces for every $C$,

$$D_C\omega = \Omega. \tag{12}$$

**Theorem 2.** (*Generalized Bianchi formula.*) *We have*

$$D_C\Omega = -\Omega \barwedge \delta(\Gamma, C, Q). \tag{13}$$

Proof. Applying absolute exterior differentiation to the structure equation and using (10) and (12), we obtain (13).

If $\Gamma$ is homogeneous, we have $D_C\Omega = 0$. We remark that the first author has deduced, [2], that for any (generalized) connection $\Gamma$ the absolute exterior differential of its curvature with respect to the vertical prolongation $V\Gamma$ of $\Gamma$ vanishes. For homogeneous connections, $V\Gamma = Q(\Gamma \oplus C)$ holds by the definition of $\delta(\Gamma, C, Q)$. This gives another explanation of the role of the Bianchi identity for homogeneous connections.

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*Authors' addresses:* I. Kolář, 603 00 Brno, Mendlovo nám. 1 (Matematický ústav ČSAV); V. Lešovský, 602 00 Brno, Botanická 54.