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INDEPENDENCE IN A SET WITH ORTHOGONALITY

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1. This paper is devoted to a study of independent sets in a set with an orthogonality relation $(\Omega, \perp)$. It is assumed that the induced complete orthomodular lattice $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ satisfies the axioms A and V given below. The independence is considered from different points of view and attention is also payd to their interrelations.

2. Let us restate here, for the convenience of the reader, some equivalent conditions on a lattice with an orthogonality relation $\mathscr{P} = (P, \leqq, \perp, 1, 0)$:

2.1. $\mathscr{P}$ is orthomodular.

2.2. If $a, b \in P$, $a \leqq b$, then $b = a \vee (a^\perp \wedge b)$.

2.3. If $a, b \in P$, $a \leqq b$, $a^\perp \wedge b = 0$, then $a = b$.

2.4. If $a, b, c \in P$, $a \leqq c$, $b \leqq c^\perp$, then $(a \vee b) \wedge c = a$.

If $(\Omega, \perp)$ is a set with an orthogonality relation and $\mathscr{S} = (S, \subset, \perp, \Omega, \{o\})$ is the corresponding complete lattice with orthogonality, we shall assume that $\mathscr{S}$ satisfies the following axioms.

2.5. Axiom A. For every $x \in \Omega$, $x \neq o$, $\{x\}^{\perp\perp}$ is an atom in $\mathscr{S}$.

2.6. Lemma. *If Axiom* A *is satisfied and if* $x, y \in \Omega$, $x \neq o \neq y$, $x \notin \{y\}^\perp$, *then* $\{x\}^{\perp\perp} \vee \{y\}^\perp = \Omega = \{x\}^\perp \vee \{y\}^{\perp\perp}$.

Proof. If $x \in \{y\}^{\perp\perp}$, then $\{x\}^{\perp\perp} = \{y\}^{\perp\perp}$ and the statement is true. Let us suppose that $x \notin \{y\}^{\perp\perp}$. If $\{x\}^\perp \cap \{y\}^{\perp\perp} \neq \{o\}$, then $\{y\}^{\perp\perp} \subset \{x\}^\perp$, hence $\{x\}^{\perp\perp} \subset \{y\}^\perp$ – a contradiction. Hence $\{x\}^\perp \cap \{y\}^{\perp\perp} = \{o\}$ and we have $\{x\}^{\perp\perp} \vee \{y\}^\perp = \Omega$. If $x \notin \{y\}^\perp$, then $y \notin \{x\}^\perp$ and therefore $\{y\}^{\perp\perp} \vee \{x\}^\perp = \Omega$. Lemma is proved.

2.7. Axiom V. If $x \in \Omega$, $A \in S$, $x \notin A$, $x \notin A^\perp$, then there exist an atom $A_1 \subset A$ and an atom $A_2 \subset A^\perp$ such that $x \in A_1 \vee A_2$.

2.8. Lemma. *If the lattice $\mathcal{S}$ is orthomodular then the atoms A_1 and A_2 from Axiom* V *are unique and the following equations are satisfied*:

$$A_1 = (\{x\}^{\perp\perp} \vee A^{\perp}) \cap A\,, \quad A_2 = (\{x\}^{\perp\perp} \vee A) \cap A^{\perp}\,.$$

Proof. If $x \in A_1 \vee A_2$ then, according to Statement 2.4, we have $(\{x\}^{\perp\perp} \vee A^{\perp}) \cap \cap A \subset (A_1 \vee A_2 \vee A^{\perp}) \cap A = A_1$. It is true that $A^{\perp} \subset \{x\}^{\perp\perp} \vee A^{\perp}$. If $(\{x\}^{\perp\perp} \vee \vee A^{\perp}) \cap A = \{o\}$ then, according to Statement 2.3, we have $\{x\}^{\perp\perp} \vee A^{\perp} = A^{\perp}$, hence $x \in A^{\perp}$ – a contradiction. Therefore, $(\{x\}^{\perp\perp} \vee A^{\perp}) \cap A = A_1$. The second assertion is proved analogously.

2.9. Definition. A set A, $\emptyset \neq A \subset \Omega$, $A \neq \{o\}$, is said to be *independent* if and only if

$$x \notin \bigvee_{y \in A - \{x\}} \{y\}^{\perp\perp}$$

for all $x \in A$. We say that the set A is *l-independent* if and only if $B^{\perp\perp} \neq A^{\perp\perp}$ for every subset $B \subset A$, $\emptyset \neq B \neq A$. We say that the set A is *L-independent* if and only if every nonempty finite subset $B \subset A$ is independent.

Throughout, we shall assume that the lattice $\mathcal{S}$ is orthomodular and it satisfies Axioms A *and* V.

We can conclude, from Lemma 2.8 in [1], that a set A is independent if and only if it is l-independent. We can also immediately conclude, from Lemma 2.9 in [1], that every independent set is L-independent.

The following theorem is a generalization of Theorem 2.11 in [1].

2.10. Theorem. *Let $A \subset \Omega$ be an independent set, $a \in \Omega$, $a \notin \bigvee_{x \in A} \{x\}^{\perp\perp}$. Then $A \cup \{a\}$ is an independent set as well.*

Proof. If the set A is a singleton, the assertion of the theorem is clear. Thus, we shall assume that the set A contains at least two points. Let us suppose that $z \in \{a\}^{\perp\perp} \vee \bigvee_{x \in A - \{z\}} \{x\}^{\perp\perp} = \{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$ for some $z \in A$. Since $a \notin \notin (A - \{z\})^{\perp\perp}$, according to Lemma 2.1 in [1] there exists $b \in \Omega$, $b \perp A - \{z\}$, such that $\{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp} = \{b\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$. It is true, by our assumption, that $\{z\}^{\perp\perp} \subset \{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$, hence $A^{\perp\perp} = \{z\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp} \subset \subset \{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$. Let us notice that $A^{\perp} \cap [\{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}] = \{z\}^{\perp} \cap \cap (A - \{z\})^{\perp} \cap [\{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}] = \{z\}^{\perp} \cap (A - \{z\})^{\perp} \cap [\{b\}^{\perp\perp} \vee (A - - \{z\})^{\perp\perp}] = \{z\}^{\perp} \cap \{b\}^{\perp\perp}$, where the last identity follows from the relation $\{b\}^{\perp\perp} \subset \subset (A - \{z\})^{\perp}$ and from Statement 2.4. However,

$$\{z\}^{\perp} \cap \{b\}^{\perp\perp} = \begin{cases} \{o\} & \text{if and only if} \quad b \notin \{z\}^{\perp} \\ \{b\}^{\perp\perp} & \text{if and only if} \quad b \in \{z\}^{\perp}\,. \end{cases}$$

If $\{z\}^{\perp} \cap \{b\}^{\perp\perp} = \{o\}$ then, by Statement 2.3, we have $A^{\perp\perp} = \{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$, hence $a \in A^{\perp\perp}$, contrary to our hypothesis If $\{z\}^\perp \cap \{b\}^{\perp\perp} = \{b\}^{\perp\perp}$ then $\{b\}^{\perp\perp} \subset \{z\}^\perp$ and, since $\{b\}^{\perp\perp} \subset (A - \{z\})^\perp$, it follows that $\{b\}^{\perp\perp} \subset \{z\}^\perp \cap (A - \{z\})^\perp = A^\perp$ as well. Since $z \notin (A - \{z\})^{\perp\perp}$ in accordance with Lemma 2.1 in [1], there exists a $c \in \Omega$, $c \neq o$, $c \perp A - \{z\}$ such that $A^{\perp\perp} = \{z\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp} = \{c\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$; hence, by Statement 2.4, we have $A^{\perp\perp} \cap \{c\}^\perp = (A - \{z\})^{\perp\perp}$. Since $b \perp A - \{z\}$, $b \perp z$, it follows that $b \perp A^{\perp\perp} = \{c\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp}$, hence $\{b\}^{\perp\perp} \subset \{c\}^\perp$. Now, we have $z \in \{a\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp} = \{b\}^{\perp\perp} \vee (A - \{z\})^{\perp\perp} = \{b\}^{\perp\perp} \vee (A^{\perp\perp} \cap \{c\}^\perp) \subset (\{b\}^{\perp\perp} \vee A^{\perp\perp}) \cap (\{b\}^{\perp\perp} \vee \{c\}^\perp) = (\{b\}^{\perp\perp} \vee A^{\perp\perp}) \cap \{c\}^\perp$. Next, $z \in \{z\}^{\perp\perp} = \{z\}^{\perp\perp} \cap A^{\perp\perp} \subset (\{b\}^{\perp\perp} \vee A^{\perp\perp}) \cap \{c\}^\perp \cap A^{\perp\perp} = A^{\perp\perp} \cap \{c\}^\perp = (A - \{z\})^{\perp\perp}$, contrary to the hypothesis. Thus, $\{z\}^\perp \cap \{b\}^{\perp\perp}$ is equal neither to $\{o\}$ nor to $\{b\}^{\perp\perp}$. Hence, the set $A \cup \{a\}$ is, in fact, independent. Theorem is proved.

2.11. Theorem. *Let $A \subset \Omega$ be an L-independent set, $a \in \Omega$, and let us suppose that, for every nonempty finite set $B \subset A$, $a \notin \bigvee_{x \in B} \{x\}^{\perp\perp}$. Then $A \cup \{a\}$ is also an L-independent set.*

Proof. The statement is true for a singleton set A. Thus, we shall assume that the set A contains at least two points. If the set $A \cup \{a\}$ is not L-independent then there is a nonempty finite set $B \subset A$ such that the set $B \cup \{a\}$ is not independent. This is only possible if there is an element $z \in B$ such that $z \in \{a\}^{\perp\perp} \vee \bigvee_{x \in B - \{z\}} \{x\}^{\perp\perp} = \{a\}^{\perp\perp} \vee (B - \{z\})^{\perp\perp}$. And now, we can follow the proof of Theorem 2.10 replacing the set A by the set B. The proof is concluded.

2.12. Theorem. *Let us suppose that $A, B \subset \Omega$, let B be a finite set with $A \cap B = \emptyset$, $A \neq \emptyset \neq B$ and let $A \cup B$ be an independent set. Then*

$$\bigcap_{x \in B} (A \cup B - \{x\})^{\perp\perp} = A^{\perp\perp}.$$

Proof. Let $x_1, x_2 \in B$, $x_1 \neq x_2$. Let us denote $C = B - \{x_1, x_2\}$. Since $x_1 \notin (A \cup C)^{\perp\perp}$, $x_2 \notin (A \cup C)^{\perp\perp}$, according to Lemma 2.1 in [1] there are $y_1, y_2 \in \Omega$, $y_1 \perp (A \cup C)^{\perp\perp}$, $y_2 \perp (A \cup C)^{\perp\perp}$ such that $(A \cup B - \{x_2\})^{\perp\perp} = (A \cup C \cup \{x_1\})^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \{x_1\}^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \{y_1\}^{\perp\perp}$, $(A \cup B - \{x_1\})^{\perp\perp} = (A \cup C \cup \{x_2\})^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \{x_2\}^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \{y_2\}^{\perp\perp}$. Since $(A \cup C)^{\perp\perp} \subset \{y_1\}^\perp$, $(A \cup C)^{\perp\perp} \subset \{y_2\}^\perp$, by Statement 2.2 we have $\{y_1\}^\perp = (A \cup C)^{\perp\perp} \vee [(A \cup C)^\perp \cap \{y_1\}^\perp]$, $\{y_2\}^\perp = (A \cup C)^{\perp\perp} \vee [(A \cup C)^\perp \cap \{y_2\}^\perp]$, hence $(A \cup C)^\perp \cap (\{y_1\}^\perp \vee \{y_2\}^\perp) = (A \cup C)^\perp \cap \{(A \cup C)^{\perp\perp} \vee [(A \cup C)^\perp \cap \{y_1\}^\perp] \vee [(A \cup C)^\perp \cap \{y_2\}^\perp]\} = ((A \cup C)^{\perp\perp} \vee \{(A \cup C)^\perp \cap [(A \cup C)^{\perp\perp} \vee \{y_1\}^{\perp\perp}] \cap [(A \cup C)^{\perp\perp} \vee \{y_2\}^{\perp\perp}]\}) = \{[(A \cup C)^{\perp\perp} \vee \{y_1\}^{\perp\perp}] \cap [(A \cup C)^{\perp\perp} \vee \{y_2\}^{\perp\perp}]\}^\perp$, where the last identity follows from Statement 2.2 and from the fact that $(A \cup C)^{\perp\perp} \subset [(A \cup C)^{\perp\perp} \vee \{y_1\}^{\perp\perp}] \cap [(A \cup C)^{\perp\perp} \vee \{y_2\}^{\perp\perp}]$. Thus, we have proved the identity $(A \cup C)^{\perp\perp} \vee (\{y_1\}^{\perp\perp} \cap \{y_2\}^{\perp\perp}) = [(A \cup C)^{\perp\perp} \vee \{y_1\}^{\perp\perp}] \cap [(A \cup C)^{\perp\perp} \vee \{y_2\}^{\perp\perp}]$. However,

$\{y_1\}^{\perp\perp} \cap \{y_2\}^{\perp\perp} = \{o\}$ because otherwise we have $\{y_1\}^{\perp\perp} = \{y_2\}^{\perp\perp}$. Therefore, $x_1 \in (A \cup C)^{\perp\perp} \vee \{x_1\}^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \{y_1\}^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \{y_2\}^{\perp\perp} = (A \cup C)^{\perp\perp} \vee \vee \{x_2\}^{\perp\perp} = (A \cup B - \{x_1\})^{\perp\perp}$, which contradicts the independence of the set $A \cup B$. Now, $\bigcap_{x \in B} (A \cup B - \{x\})^{\perp\perp} = \bigcap_{x \in A \cup B - \{x_1\}} (A \cup B - \{x_1\})^{\perp\perp} \cap (A \cup B - \{x\})^{\perp\perp} = \bigcap_{x \in A \cup B - \{x_1\}} (A \cup B - \{x_1\} - \{x\})$ for all $x_1 \in B$. The statement of the theorem can be proved by induction.

3. Let $\mathscr{A} = \{A_i : i \in I\}$ be a chain of L-independent sets in Ω. Let us denote $A = \bigcup_{i \in I} A_i$. Let B be a finite set with $\emptyset \neq B \subset A$. Then there exists an $i_0 \in I$ such that $B \subset A_{i_0}$. Therefore, B is an independent set. In accordance with Zorn's lemma, there exist maximal L-independent sets in Ω with respect to the set-theoretical inclusion.

If $A \subset \Omega$ is a maximal L-independent set, then $\bigvee_{x \in A} \{x\}^{\perp\perp} = \Omega$. Indeed, otherwise we have $y \in \Omega$ such that $y \notin \bigvee_{x \in A} \{x\}^{\perp\perp}$. It follows, in this case, that $y \notin \bigvee_{x \in B} \{x\}^{\perp\perp}$ for every non-empty finite subset $B \subset A$. According to Theorem 2.11, the set $A \cup \{y\}$ is L-independent, consequently, the set A is not a maximal L-independent set. This contradiction concludes the proof of our assertion.

3.1. Corollary. *Let $A \subset \Omega$ be a maximal L-independent set. Then for every $x \in \Omega$ there is a finite set $B_x \subset A$ such that*

$$x \in B_x^{\perp\perp} = \bigvee_{y \in B_x} \{y\}^{\perp\perp} .$$

Proof. If $x \notin B^{\perp\perp}$ for every finite $B \subset A$ then, according to Theorem 2.11, the set $A \cup \{x\}$ is L-independent, which contradicts the maximality of the set A.

3.2. Lemma. *If $A \subset \Omega$ is a maximal L-independent set, then for every $x \in \Omega$, $x \neq o$, there exists a smallest set $B_x \subset A$, with respect to the inclusion, such that*

$$x \in B_x^{\perp\perp} = \bigvee_{y \in B_x} \{y\}^{\perp\perp} .$$

Proof. If $x \in B_x^{\perp\perp}$, $x \in C_x^{\perp\perp}$, $B_x \subset A$, $C_x \subset A$ and both B_x and C_x are finite, then $B_x \cap C_x \neq \emptyset$. If this is not true, i.e., if $B_x \cap C_x = \emptyset$ then, according to Theorem 2.12, we have $B_x^{\perp\perp} = \bigcap_{y \in C_x} (B_x \cup C_x - \{y\})^{\perp\perp}$, $C_x^{\perp\perp} = \bigcap_{z \in B_x} (B_x \cup C_x - \{z\})^{\perp\perp}$ and $B_x^{\perp\perp} \cap \cap C_x^{\perp\perp} = (B_x \cup C_x - \{v\})^{\perp\perp} \cap \bigcap_{y \in B_x \cup C_x - \{v\}} (B_x \cup C_x - \{y\})^{\perp\perp} = (B_x \cup C_x - \{v\})^{\perp\perp} \cap \cap \{v\}^{\perp\perp} = \{o\}$ for an element $v \in B_x \cup C_x$. Thus, we have $B_x^{\perp\perp} \cap C_x^{\perp\perp} = \{o\}$ – a contradiction. Let us denote $B_x \cap C_x = D_x$ and let us express B_x and C_x as the following disjoint unions: $B_x = D_x \cup E_x$, $C_x = D_x \cup F_x$. According to Theorem

2.12, we have $B_x^{\perp\perp} = \bigcap_{y \in F_x} (D_x \cup E_x \cup F_x - \{y\})^{\perp\perp}$, $C_x^{\perp\perp} = \bigcap_{z \in E_x} (D_x \cup E_x \cup F_x - \{z\})^{\perp\perp}$; hence $B_x^{\perp\perp} \cap C^{\perp\perp} = \bigcap_{y \in E_x \cup F_x} (D_x \cup E_x \cup F_x - \{y\})^{\perp\perp} = D_x^{\perp\perp}$, which yields $x \in D_x^{\perp\perp}$.

3.3. Theorem. *Let $M, N \subset \Omega$ be maximal L-independent sets. Then* card M = card N.

Proof. We can assume, without loss of generality, that $M \cap N = \emptyset$.

a) To each element $x \in M$ assign an element $y = \varphi(x) \in N$ such that the set $\{x\} \cup (N - \{y\})$ is L-independent. We shall prove that such an element y exists. In accordance with Lemma 3.2, there exists a minimal finite set $B_x \subset N$ such that $x \in B_x^{\perp\perp} = \bigvee_{y \in B_x} \{y\}^{\perp\perp}$. Let us choose any element $y \in B_x$. If $B \subset N - \{y\}$, $B \neq \emptyset$ is a finite set then $x \notin B^{\perp\perp}$ because otherwise we have $B_x \subset B$, hence $y \in B$ – a contradiction. It follows from Theorem 2.11 that $\{x\} \cup (N - \{y\})$ is L-independent.

b) Let $\mathscr{F}$ be a family of functions such that the following assertions hold for each $f \in \mathscr{F}$:

(i) $D_f \subset M$, $R_f \subset N$, where D_f is the domain of f and R_f is the range of f;
(ii) $f : D_f \to R_f$ is a bijective mapping;
(iii) $D_f \cup (N - R_f)$ is an L-independent set.

We shall order the family $\mathscr{F}$ as follows: $f \leqq g$ if and only if $D_f \subset D_g$ and $f(x) = g(x)$ for all $x \in D_f$.

There is a nonempty chain $\mathscr{F}_0 \subset \mathscr{F}$ containing the function φ which we have constructed in part a) of this proof. Every nonempty chain $\mathscr{F}_0 \subset \mathscr{F}$ has an upper bound f_0 in $\mathscr{F}$. Indeed, put $D_{f_0} = \bigcup_{f \in \mathscr{F}_0} D_f$ and $f_0(x) = f(x)$ for each $x \in D_f$ and each $f \in \mathscr{F}_0$. The function f_0 satisfies the conditions (i) and (ii), where $R_{f_0} = \bigcup_{f \in \mathscr{F}_0} R_f$. We shall prove that it satisfies the condition (iii) as well; therefore, $f_0 \in \mathscr{F}$. For every $f \in \mathscr{F}_0$ the set $D_f \cup (N - R_f)$ is L-independent, consequently, for every $f \in \mathscr{F}_0$, the set $D_f \cup (N - \bigcup_{g \in \mathscr{F}_0} R_g) = D_f \cup (N - R_{f_0})$ is L-independent as well. Now, the family $\{D_f \cup (N - R_{f_0}) : f \in \mathscr{F}_0\}$ is a chain of L-independent sets. As is shown at the beginning of part 3, the set $\bigcup_{f \in \mathscr{F}_0} \{D_f \cup (N - R_{f_0})\} = D_{f_0} \cup (N - R_{f_0})$ is L-independent. Therefore, in accordance with Zorn's lemma, there is a maximal element h in $\mathscr{F}$.

c) We shall prove that $R_h = N$. If not, then $R_h \neq N$. It follows that $D_h \neq M$ because otherwise the set $D_h \cup (N - R_h) = M \cup (N - R_h)$ is L-independent, which is not true. Choose $u \in M - D_h$ and $v \in N - R_h$. If $u \notin B^{\perp\perp}$ for every finite $B \subset D_h \cup (N - R_h)$ we put $h'(u) = v$ and $h'(x) = h(x)$ for all $x \in D_h$. The extension h' of the function h satisfies the conditions (i) and (ii). According to Theorem 2.11, the set $D_h \cup \{u\} \cup (N - R_h)$ is L-independent, hence, the set $D_h \cup \{u\} \cup (N - R_h - \{v\})$ is also L-independent. Consequently, the extension h' satisfies the condition (iii), therefore the function h is not the maximal element in $\mathscr{F}$. If $u \in B^{\perp\perp}$ for a finite set $B \subset D_h \cup (N - R_h)$ then, by the same argument as in the proof of Lemma 3.2, we can see that there is a smallest set $B_u \subset B$ such that $u \in B_u^{\perp\perp}$. Of course, $B_u \cap (N - R_h) \neq \emptyset$. Indeed, otherwise we have $B_u \subset D_h \subset M$, therefore the set M is not L-independent. We chose $t \in B_u \cap (N - R_h)$ and put $h''(u) = t$. The extension h'' of the function h satisfies the conditions (i) and (ii). Let $C \subset D_h \cup (N - R_h - \{t\})$ be a finite set, $C \neq \emptyset$. Then $u \notin C^{\perp\perp}$, because otherwise we have $B_u \subset C$, hence $t \in C$ — a contradiction. According to Theorem 2.11 the set $D_h \cup \{u\} \cup (N - R_h - \{t\})$ is L-independent. Thus, the extension h'' satisfies the condition (iii) as well, which implies that the function h is not the maximal element in $\mathscr{F}$. We can conclude that $R_h = N$.

d) The inverse mapping $\psi = h^{-1} : N \to D_h \subset M$ is a one-to-one mapping of N to M. In view of the symmetry of the sets M and N, there is a one-to-one mapping of M to N as well. According to the well known Cantor-Bernstein Theorem, we have card M = card N. The theorem is proved.

Literature

[1] *Havrda J.:* Gram-Schmidt's orthogonalization based on the concept of generalized orthogonality, Čas. pěst. mat. *106* (1981), 335—346.

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