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ON THE MAXIMUM NUMBER OF ARCS
IN SOME CLASSES OF GRAPHS

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INTRODUCTION AND NOTATION

Under an oriented graph \( G(X, U) \) we always understand a directed graph without loops and 2-cycles, with the set of points \( X \) and set of arcs \( U \). If \(|X| = p, |U| = q\), we also write \( G(p, q) \). In such a graph, \( d_G(x) \), for \( x \in X \), denotes the sum of the out-degree and in-degree of \( x \) and \( \delta(G) = \min \{d_G(x); x \in X\} \).

We shall also denote, for a real \( t \), by \( \lceil t \rceil \) the integer satisfying \( t \leq \lceil t \rceil < t + 1 \), by \( \lfloor t \rfloor \) the integer satisfying \( t - 1 < \lfloor t \rfloor \leq t \).

1. CALCULATION OF \( f_1(p) \)

We say that an oriented graph \( G(p, q) \) satisfies property \( (P_1) \) if for all pairs of points \( x \) and \( y \), there exists at most one path from \( x \) to \( y \), and we let \( \mathcal{G}_1 \) be the set of all oriented graphs \( G(p, q) \) satisfying property \( (P_1) \). Let \( f_1(p) = \max \{q; G(p, q) \in \mathcal{G}_1\} \).

**Theorem 1.** \( f_1(p) = \lfloor \frac{1}{4} p^2 \rfloor \) for \( p \geq 4 \).

**Proof.** We first note some properties of the graphs \( G(p, q) \) belonging to \( \mathcal{G}_1 \).

(a) If \([x_1, x_2, \ldots, x_r]\) is a directed path of \( G = (X, U) \) and if \( u \in X \setminus \{x_1, x_2, \ldots, x_r\} \), then \( u \) is joined by an arc.
(1) to at most one \( x_i \), \( i = 1, 2, \ldots, r \) and
(2) from at most one \( x_i \), \( i = 1, 2, \ldots, r \).

If \([x_1, x_2, \ldots, x_r, x_1]\) is a cycle, then \( u \) is joined by an arc (of any orientation) with at most one \( x_i \) altogether.

(b) If \( C \) is a cycle in \( G \in \mathcal{G}_1 \) of length \( n \geq 3 \), and if \( G' \) is the graph obtained from \( G \) by contracting the points of \( C \) to a point \( v \), with \( v \) joined to (from) a point \( u \) if any point of \( C \) is joined to (from) \( u \), then \( G' \) also belongs to \( \mathcal{G}_1 \).

Now, consider the only two possible cases:

(i) The graph \( G \) contains a triangle with points \( x, y \) and \( z \). The subgraph induced by \( \{x, y, z\} \) is necessarily a 3-cycle. Let \( G'(p - 2, q - 3) \) be the graph obtained by contracting this cycle as above; this gives \( q - 3 \leq f_1(p - 2) \) and so \( q \leq f_1(p - 2) + 3 \).

(ii) The graph \( G \) does not contain a triangle and since \( G \) is antisymmetric then \( q \leq |\frac{1}{4}p^2| \) (Turán). We deduce that

\[
f_1(p) = \max\left(\left\lfloor \frac{p^2}{4} \right\rfloor; f_1(p - 2) + 3\right).
\]

Now, since \( f_1(2) = 1 \) and \( f_1(3) = 3 \), we get that

\[
f_1(p) \leq \left\lfloor \frac{p^2}{4} \right\rfloor \text{ for all } p \geq 4.
\]

The value \( \left\lfloor \frac{1}{4}p^2 \right\rfloor \) is attained for the complete bipartite graph \((A, B, U)\) where \( |A| = \left\lfloor \frac{1}{2}p \right\rfloor \) and \( |B| = \left\lfloor \frac{1}{2}p \right\rfloor \) with arcs oriented from \( A \) to \( B \). So

\[
f_1(p) = \left\lfloor \frac{p^2}{4} \right\rfloor \text{ for all } p \geq 4.
\]

Remark. We note that if \( G(p, q) \) is a directed graph (possible with 2-cycles) then

\[
f_1(p) = 2p - 2 \text{ for all } p \leq 7
\]

and

\[
f_1(p) = \left\lfloor \frac{1}{4}p^2 \right\rfloor \text{ for all } p \geq 7.
\]

2. CALCULATION OF \( f_2(p) \)

We say that an oriented graph \( G(p, q) \) has property \((P_2)\) if, for all pairs of points \( x \) and \( y \) of \( G \), there are at most two distinct directed paths from \( x \) to \( y \) and we denote by \( \mathcal{G}_2 \) the set of all oriented graphs \( G(p, q) \) with property \((P_2)\). We let

\[
f_2(p) = \max\{q; G(p, q) \in \mathcal{G}_2\}.
\]
Theorem 2. \( f_2(p) = \lfloor \frac{1}{2}(p - 1) \rfloor + \lfloor \frac{1}{4}p^2 \rfloor \) for all \( p \geq 4 \).

Proof. We shall say that a graph \( G(p, q) \) satisfies the relation \((R)\) if \( q \leq \lfloor \frac{1}{2}(p - 1) \rfloor + \lfloor \frac{1}{4}p^2 \rfloor \).

We shall first establish the following two lemmas.

Lemma 1. Let \( G(p, q) \) be a graph having a point \( x \) such that \( d_G(x) \leq \lfloor \frac{1}{4}p \rfloor \), then \( G(p, q) \) satisfies the relation \((R)\) if the graph \( G' = G \setminus \{x\} \) obtained by deleting the point \( x \) and all arcs adjacent with \( x \) satisfies the relation \((R)\).

Proof. We have

\[
q - d_G(x) \leq \left\lfloor \frac{p - 2}{2} \right\rfloor + \left\lfloor \frac{(p - 1)^2}{4} \right\rfloor,
\]

which yields

\[
q \leq \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p - 2}{2} \right\rfloor + \left\lfloor \frac{(p - 1)^2}{4} \right\rfloor.
\]

Consequently, it is enough to show by inspection that

\[
\left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{p - 2}{2} \right\rfloor + \left\lfloor \frac{(p - 1)^2}{4} \right\rfloor \leq \left\lfloor \frac{p - 1}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor.
\]

Lemma 2. For every graph \( G(p, q) \in \mathcal{G}_2 \) we have \( \delta(G) \leq \lfloor \frac{1}{4}p \rfloor \).

Proof. If not then there exists a graph \( G_0 \in \mathcal{G}_2 \) such that \( \delta(G_0) \geq \lfloor \frac{1}{4}p \rfloor + 1 \) and so, for any two points \( x \) and \( y \) of \( G_0 \) joined by an arc,

\[
d_{G_0}(x) + d_{G_0}(y) \geq 2 \left\lfloor \frac{p}{2} \right\rfloor + 2 \geq p + 2.
\]

This implies the existence of at least two points of \( X \setminus \{x, y\} \) simultaneously joined by arcs with \( x \) and \( y \). We shall use this fact to show that all triangles in \( G_0 \) are 3-cycles (and deduce a contradiction). If not, without loss of generality, let \( u, v, w \) be the points of a triangle formed by the arc \([u, v]\) and the directed path \([u, w, v]\). By the above observation there is a point \( z \), different from \( w \), joined by arcs of any orientation with \( u \) and \( v \). The triangle induced by \( \{u, v, z\} \) is necessarily a 3-cycle (Fig. 1) by \((P_2)\). We note that \( w \) and \( z \) cannot be joined by an arc by \((P_2)\) so that

![Fig. 1.](image-url)
there is a point \( s \) different from \( w \), other than \( u \), which is joined by arcs with both points \( v \) and \( z \). All the four possible orientations of the edges \((v, s)\) and \((s, z)\) lead to three or more distinct directed paths from a point in \( G_0 \) to another point (see Fig. 2 where \( \square \) marks the starting point of three or more distinct paths to the point marked \( \bigcirc \)). Now, to any arc \([u, v]\) of \( G_0 \in \mathcal{G}_2 \) correspond two distinct paths of length two from \( v \) to \( u \) in \( G_0 \), say \([v, w, u]\) and \([v, z, u]\) (see Fig. 3). But since \( w \) cannot be joined by an arc to (or from) \( z \) by \((P_2)\), there exists another point different from \( u \), say \( s \), joined by arcs with both \( v \) and \( z \), forming the 3-cycle \([v, z, s, v]\) (Fig. 4). Similarly, since \( u \) cannot be joined with \( s \) and \( w \) cannot be joined with \( z \), there exists another point different from \( v \), say \( r \), joined by arcs with both \( z \) and \( s \), forming the 3-cycle \([z, s, r, z]\) (Fig. 4). This leads, however, to three distinct directed paths from \( s \) to \( u \) contradicting \((P_2)\). The proof is complete.

Now we are able to finish the proof of theorem 2. We shall use induction to show first that all graphs in \( \mathcal{G}_2 \) satisfy the relation \((R)\). It is easy to see that all \( G(p, q) \in \mathcal{G}_2 \) of order \( p \leq 3 \) satisfy \((R)\). Assume that \( n > 3 \) and that the assertion is true for all graphs \( G(p, q) \in \mathcal{G}_2 \) such that \( p \leq n - 1 \). Let \( G(n, q) \in \mathcal{G}_2 \). By lemma 2, there exists a point \( x \) in \( G \) such that \( d_G(x) \leq \left\lfloor \frac{n}{2} \right\rfloor \). Since \( G' = G \setminus \{x\} \) belongs to \( \mathcal{G}_2 \), it satisfies \((R)\) by the induction hypothesis. Therefore, \( G \) satisfies \((R)\) by lemma 1, i.e. for all \( G(p, q) \in \mathcal{G}_2, \ q \leq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor \). Hence

\[
f_2(p) \leq \left\lfloor \frac{p - 1}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor.
\]
Finally, the complete tripartite graph $(A, B, C, U)$ where $|A| = \lfloor \frac{1}{4}(p - 1) \rfloor$, $B = \lfloor \frac{1}{4}(p - 1) \rfloor$, $|C| = 1$, with orientation from $A$ to $B$, from $A$ to $C$ and from $C$ to $B$ belongs to $\mathcal{G}_2$ and

$$|U| = q = \left\lfloor \frac{p - 1}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor.$$ 

Hence

$$f_2(p) = \left\lfloor \frac{p - 1}{2} \right\rfloor + \left\lfloor \frac{p^2}{4} \right\rfloor \text{ for } p \geq 4.$$ 

References


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