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A FUNCTION WHICH PRESERVES CONNECTED SPACES

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1. INTRODUCTION

In [2] N. Levine introduced the notion of semi-open sets and semi-continuity in topological spaces. S. G. Crossley and S. K. Hildebrand [1] introduced the notion of semi-homeomorphisms and investigated topological properties which are preserved under such functions. In [1], among others, they showed that (1) connected spaces are preserved under semi-homeomorphisms, (2) the images of connected sets under a semi-homeomorphism are not necessarily connected and (3) the images of open connected sets under a semi-homeomorphism are connected. P. E. Long and D. A. Carnahan [3] showed that connected spaces are preserved under almost-continuous (in the sense of Singal [6]) surjections. Moreover, P. E. Long and L. L. Herrington [4] stated that the images of open connected sets are connected under open almost-continuous functions. The purpose of the present note is to introduce a weak form of continuity, called strongly semi-continuous, which is stronger than semi-continuity due to N. Levine and to show that the image of an open connected set under a strongly semi-continuous function is connected.

Throughout the present note, $X$ and $Y$ will denote topological spaces on which no separation axioms are assumed, and a function $f$ of $X$ into $Y$ will be denoted by $f : X \rightarrow Y$. Let $S$ be a subset of $X$. The closure (resp. interior) of $S$ in $X$ will be denoted by $\text{Cl}_X(S)$ (resp. $\text{Int}_X(S)$). A set $S$ of $X$ is said to be semi-open [2] (resp. an $\alpha$-set [5]) if $S \subset \text{Cl}_X(\text{Int}_X(S))$ (resp. $S \subset \text{Int}_X(\text{Cl}_X(\text{Int}_X(S)))$). The family of all semi-open sets (resp. $\alpha$-sets) of $X$ will be denoted by $SO(X)$ (resp. $\alpha(X)$).

2. STRONGLY SEMI-CONTINUOUS FUNCTIONS

Definition 2.1. A function $f : X \rightarrow Y$ is said to be strongly semi-continuous (resp. semi-continuous [2]) if $f^{-1}(V) \in \alpha(X)$ (resp. $f^{-1}(V) \in SO(X)$) for every open set $V$ of $Y$.

Strongly semi-continuous functions will be denoted by s.s.c. functions.
Definition 2.2. A function \( f : X \to Y \) is said to be \textit{almost-continuous} \([6]\) if for each \( x \in X \) and each open set \( V \) containing \( f(x) \), there exists an open set \( U \) containing \( x \) such that \( f(U) \subseteq \text{Int}_Y(\text{Cl}_Y(V)) \).

It is obvious that continuity implies strong semi-continuity and strong semi-continuity implies semi-continuity. However, the converses are not necessarily true as the following two examples show.

Example 2.3. Let \( X = \{a, b, c\} \), \( \Gamma = \emptyset, \{a\}, \{a, b\}, \{a, c\}, X \) and \( \Gamma^* = \emptyset, \{a\}, \{a, b\}, \{a, c\}, X \). Then, the identity function \( f : (X, \Gamma) \to (X, \Gamma^*) \) is s.s.c. but not continuous.

Example 2.4. Let \( X = \{a, b, c\} \), \( \Gamma = \emptyset, \{a\}, \{b\}, \{a, b\}, X \) and \( \Gamma^* = \emptyset, \{a\}, \{b\}, \{a, b\}, X \). Then, the identity function \( f : (X, \Gamma) \to (X, \Gamma^*) \) is semi-continuous but not s.s.c.

Definition 2.5. A function \( f : X \to Y \) is said to be a \textit{semi-homeomorphism} \([1]\) if (1) \( f \) is bijective, (2) \( f^{-1}(V) \in SO(X) \) for every \( V \in SO(Y) \) and (3) \( f(U) \in SO(Y) \) for every \( U \in SO(X) \).

In Example 2.3 \( f \) is a semi-homeomorphism because it is bijective and \( SO(X, \Gamma) = \emptyset = SO(X, \Gamma^*) \). Therefore, a semi-homeomorphism need not imply continuity. Moreover, as the following example shows, a continuous function need not be a semi-homeomorphism and hence neither an almost-continuous function nor a s.s.c. function need be a semi-homeomorphism.

Example 2.6. Let \( X = \{a, b, c, d\} \), \( \Gamma = \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X \) and \( \Gamma^* = \emptyset, \{a\}, \{a, c\}, X \). Then, the identity function \( f : (X, \Gamma) \to (X, \Gamma^*) \) is continuous but not a semi-homeomorphism.

3. CONNECTED SPACES

It is well known that connected spaces are preserved under continuous surjections. It is also known that connected spaces are preserved under semi-homeomorphisms \([1, \text{Theorem 2.12}]\) or almost-continuous surjections \([3, \text{Theorem 4}]\). Similarly, we have

Theorem 3.1. If \( X \) is connected and \( f : X \to Y \) is a s.s.c. surjection, then \( Y \) is connected.

Proof. Suppose that \( Y \) is not connected. Then, there exist nonempty open sets \( V_1 \) and \( V_2 \) of \( Y \) such that \( V_1 \cup V_2 = Y \) and \( V_1 \cap V_2 = \emptyset \); hence \( f^{-1}(V_1) \cup f^{-1}(V_2) = X \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \). Since \( f \) is s.s.c. and surjective, we have \( \emptyset \neq f^{-1}(V_j) \subseteq \text{Int}_X(\text{Cl}_X(\text{Int}_X(f^{-1}(V_j)))) \) for \( j = 1, 2 \). Now, put \( U_j = \text{Int}_X(\text{Cl}_X(\text{Int}_X(f^{-1}(V_j)))) \).
for \( j = 1, 2 \), then \( U_j \) is a nonempty open set of \( X \) and \( U_1 \cap U_2 = X \). Since \( f^{-1}(V_1) \) and \( f^{-1}(V_2) \) are disjoint, we obtain \( \text{Int}_X(f^{-1}(V_1)) \cap \text{Cl}_X(\text{Int}_X(f^{-1}(V_2))) = \emptyset \) and hence \( \text{Int}_X(f^{-1}(V_1)) \cap U_2 = \emptyset \). Consequently, we obtain \( U_1 \cap U_2 = \emptyset \) which implies that \( X \) is not connected. This is a contradiction.

**Remark 3.2.** In Example 2.4 \((X, f)\) is connected and \( f \) is a semi-continuous surjection but \((X, f^*)\) is not connected. Therefore, the condition “s.s.c.” on \( f \) in Theorem 3.1 cannot be replaced by “semi-continuous”.

We recall that a function \( f : X \to Y \) is said to be connected if the image \( f(C) \) is connected for every connected set \( C \) of \( X \). It is well known that every continuous function is connected but not conversely. It is shown that semi-homeomorphisms are not connected functions [1, Example 1.5]. However, it is known that the images of open connected sets are connected under semi-homeomorphisms [1, Theorem 2.14] or open almost-continuous surjections [4, Theorem 6]. We shall show that the images of open connected sets are connected under s.s.c. surjections.

**Lemma 3.3.** If \( U \) is an open set of \( X \) and \( A \in \mathcal{A}(X) \), then \( U \cap A \in \mathcal{A}(X) \).

**Proof.** We obtain \( U \cap A \in \mathcal{A}(X) \) from

\[
\text{Int}_X(\text{Cl}_X(U \cap A)) \supseteq \text{Int}_X(U \cap \text{Cl}_X(\text{Int}_X(A))) \supseteq U \cap A.
\]

**Lemma 3.4.** Let \( U \) be an open set of \( X \) and \( A \) a subset of \( U \). Then, \( A \in \mathcal{A}(X) \) if and only if \( A \in \mathcal{A}(U) \).

**Proof.** Since \( U \) is open in \( X \), \( \text{Int}_X(B) = \text{Int}_U(B) \) for every subset \( B \) of \( U \). Thus, we have \( \text{Int}_U(\text{Cl}_U(\text{Int}_U(A))) = \text{Int}_X(\text{Cl}_X(\text{Int}_X(A))) \cap U \) which completes the proof.

**Lemma 3.5.** If \( U \) is an open set of \( X \) and \( f : X \to Y \) is s.s.c., then a function \( f_U : U \to f(U) \), defined by \( f_U(x) = f(x) \) for every \( x \in U \), is s.s.c.

**Proof.** Let \( V_U \) be any open set of a subspace \( f(U) \). Then, there exists an open set \( V \) of \( Y \) such that \( V_U = V \cap f(U) \). Since \( f \) is s.s.c., \( f^{-1}(V) \in \mathcal{A}(X) \) and hence, by Lemma 3.3 \( f_U^{-1}(V_U) = f^{-1}(V) \cap U \in \mathcal{A}(X) \) because \( U \) is open in \( X \). Therefore, by Lemma 3.4 we obtain \( f_U^{-1}(V_U) \in \mathcal{A}(U) \) which shows that \( f_U \) is s.s.c.

**Theorem 3.6.** If \( f : X \to Y \) is s.s.c., then \( f(U) \) is connected for every open connected set \( U \) of \( X \).

**Proof.** Suppose that \( f : X \to Y \) is s.s.c. and \( U \) is an open connected set of \( X \). Then, by Lemma 3.5 \( f_U : U \to f(U) \) is s.s.c. and hence, by Theorem 3.1 \( f_U(U) = f(U) \) is connected.

**Remark 3.7.** In Example 2.3 \( f \) is s.s.c. and a subset \( \{b, c\} \) is closed connected in
$(X, \mathcal{I})$ but $f(\{b, c\})$ is not connected. Therefore the condition "open" on $U$ in Theorem 3.6 can not be replaced by "closed". This shows that a s.s.c. function is not necessarily connected.

**Remark 3.8.** The inverse function $f^{-1} : (X, \mathcal{I}^*) \to (X, \mathcal{I})$ of $f$ in Example 2.4 is connected but not semi-continuous and hence not s.s.c. This shows that connected function is not necessarily s.s.c. Moreover, we observe that the converse to Theorem 3.6 is not true in general.

**Remark 3.9.** The condition "s.s.c." on $f$ in Theorem 3.6 can not be replaced by "semi-continuous" as we have noted in Remark 3.2.

**References**


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