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AN EXTREMAL PROBLEM FOR SOME CLASSES OF ORIENTED GRAPHS

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INTRODUCTION AND NOTATION

Let \( \mathcal{G}_k \) be the set of oriented graphs (directed graphs with no 2-cycle) satisfying property \( (P_k) \): For all couples of points \( x \) and \( y \), there exist at most \( k \) distinct directed paths from \( x \) to \( y \). In previous paper the authors gave the values of \( f_k(p) = \max \{ q \mid G(p, q) \in \mathcal{G}_k \} \) for \( k = 1, 2 \) where \( G(p, q) \) denotes a graph with \( p \) points and \( q \) arcs. We shall give here the value of \( f_3(p) \) and characterize those graphs in \( \mathcal{G}_k \) which have the maximum number \( f_3(p) \) of arcs for \( k = 1, 2, 3 \).

Under an oriented graph \( G(X, U) \) we shall understand a directed graph without loops and 2-cycles, with the set of points \( X \) and set of arcs \( U \). If \( |X| = p, |U| = q \), we also write \( G(p, q) \). In such a graph, \( d_G(x) \), for \( x \in X \), denotes the sum of the out-degree and in-degree of \( x \) and \( \delta(G) = \min \{ d_G(x) \mid x \in X \} \).

Arcs will be denoted by \( (u, v) \) etc., non-directed paths by \( [u, v, w] \) etc.

We shall also denote, for a real \( t \), by \( \lceil t \rceil \) the integer satisfying \( t \leq \lceil t \rceil < t + 1 \), by \( \lfloor t \rfloor \) the integer satisfying \( t - 1 < \lfloor t \rfloor \leq t \).

In section 1, we shall say that a graph \( G(p, q) \) satisfies the relation \( (R) \) if

\[
q \leq 2(p - 2) + \lfloor \sqrt{(p - 2)^2} \rfloor.
\]

1. EVALUATION OF \( f_3(p) \)

**Theorem 1.1.** \( f_3(p) = 2(p - 2) + \lfloor \sqrt{(p - 2)^2} \rfloor \) for \( p \geq 6 \).

**Proof.** We shall first establish two lemmas.

**Lemma 1.** For every graph \( G(p, q) \in \mathcal{G}_3 \) we have \( \delta(G) \leq \lceil \sqrt{p + 1} \rceil \).

**Proof.** If not then there exists a graph \( G_0 \in \mathcal{G}_3 \) such that \( \delta(G_0) \geq \lceil \sqrt{p + 1} \rceil + 1 \). Therefore, for any two adjacent points \( x \) and \( y \) of \( G_0 \),

\[
d_G(x) + d_G(y) \geq 2\lceil \sqrt{p + 1} \rceil + 2 \geq p + 3.
\]
This implies the existence of at least three points of $X \setminus \{x, y\}$ adjacent simultaneously to $x$ and $y$. We shall use this fact to show that all triangles in $G_0$ are 3-cycles and obtain a contradiction. Suppose this is not true and let $u, v, w$ be points of $X \setminus \{x, y\}$. The figure illustrates the possible arrangements of $u, v, w$ and $x, y$. The diagrams show that there are at least three points adjacent simultaneously to $x$ and $y$. This contradicts the assumption that all triangles in $G_0$ are 3-cycles. For this reason, we can conclude that $G_0$ contains a triangle that is not a 3-cycle.
a triangle formed by the arc \((u, v)\) and the (yet non-oriented) path \([u, w, v]\). By the above observation, there exist two points \(y\) and \(z\), different from \(w\), adjacent to or from \(u\) and \(v\). All the six possible orientations of the edges \([u, y]\), \([u, z]\), \([v, y]\) and \([v, z]\) are given in Fig. 1. We shall show that all these graphs are "forbidden" subgraphs in \(G_0\). It suffices to prove that the graphs in Fig. 1-a, 1-d and the graphs in Fig. 2-a and 2-b (subgraphs of 1-b, 1-c, 1-e and 1-f) are forbidden subgraphs in \(G_0\).

Let us consider the graph in Fig. 1-a. It is easy to see that \(z\) can be adjacent, to or from, neither \(w\) nor \(y\) by property \((P_3)\). Thus there is a point \(s\) different from \(u, y\) and \(w\) which is adjacent to or from both \(v\) and \(z\). All the four possible orientations of the edges \([v, s]\) and \([z, s]\) create four or more distinct paths from a point in \(G_0\) to another point (see Fig. 3 where \(X\) marks the starting point of four or more paths to the point marked \(O\)).

For the subgraph shown in Fig. 1-d, we see that \(y\) and \(z\) cannot be adjacent by \((P_3)\) and since \(d_{G_0}(y) + d_{G_0}(z) \geq p + 3\), there are at least five points in \(X \setminus \{y, z\}\) which are simultaneously adjacent to or from both \(y\) and \(z\). Thus there is a point \(s\) different from \(u, v, w\) which is adjacent to or from both \(y\) and \(z\). However, all the four possible orientations of the edges \([y, s]\) and \([z, s]\) lead to four or more distinct paths from a point to another (see Fig. 4 where again \(X\) marks the starting point of four or more paths to the point marked \(O\)).
Now for each of the subgraphs given in Fig. 2-a and Fig. 2-b there is at least one point $s$ different from $h$ and $i$, adjacent to or from $g$ and $j$. The only possible orientations of the edges $[g, s]$ and $[j, s]$ are from $j$ to $s$ and from $s$ to $g$ (see Fig. 5). It is easily seen that $s$ can be adjacent neither to or from $h$ nor to or from $i$. Thus there exists at least one point $r$ different from $g$, $h$ and $i$ which is simultaneously adjacent to or from $j$ and $s$. All the four possible orientations of $[j, r]$ and $[s, r]$ lead, however, to four or more distinct paths in $G_0$ (see Fig. 6 which corresponds to Fig. 5-a; a similar set of figures may be given for Fig. 5-b).
It follows that every triangle in $G_0$ is a 3-cycle. Hence to any arc $(w, v)$ of $G_0 \in \mathcal{G}_3$ there correspond three distinct paths of length two from $v$ to $u$ in $G_0$, say $[v, w, u]$, $[v, y, u]$ and $[v, z, u]$ (see Fig. 7). However, since $z$ cannot be adjacent to or from $y$ or $w$ by $(P_3)$, there exist at least two points $r$ and $s$ different from $u$ and adjacent simultaneously to or from $v$ and $z$. The induced orientation on the new edges leads then to more than three paths from $z$ to $w$, in contradiction to $(P_3)$. The lemma is proved.
Lemma 2. Let a graph $G(p, q)$ have a point $x$ such that $d_{G}(x) \leq \lceil \frac{1}{4}(p + 1) \rceil$. Then $G(p, q)$ satisfies the relation (R) if $G' = G \setminus \{x\}$ (obtained from $G$ by omitting the point $x$ and the arcs incident with $x$) satisfies the relation (R).

Proof. Let the relation (R) for $G'$ be fulfilled:

$$q - d_{G}(x) \leq 2(p - 3) + \left\lfloor \frac{1}{4}(p - 3)^2 \right\rfloor.$$

Then

$$q \leq \left\lfloor \frac{1}{4}(p + 1) \right\rfloor + 2(p - 3) + \left\lfloor \frac{1}{4}(p - 3)^2 \right\rfloor.$$

However, by inspecting the four cases mod 4 it follows easily that

$$\left\lfloor \frac{1}{4}(p + 1) \right\rfloor + 2(p - 3) + \left\lfloor \frac{1}{4}(p - 3)^2 \right\rfloor = 2(p - 2) + \left\lfloor \frac{1}{4}(p - 2)^2 \right\rfloor$$

so that $G(p, q)$ satisfies (R).

To finish the proof of theorem 1.1, we shall show by induction that all graphs in $\mathcal{G}_3$ satisfy the relation (R).

Let us show first that any graph $G(6, q) \in \mathcal{G}_3$ satisfies (R), i.e. $q \leq 12$. Suppose there is a graph $G_0(6, q) \in \mathcal{G}_3$ such that $q \geq 13$. By lemma 1, there exists a point $x$ in $G_0$ such that $\delta(G_0) = d_{G_0}(x) \leq 4$. Let $G'_0(5, q') = G_0(6, q) \setminus \{x\}$. Since $G'_0 \in \mathcal{G}_3$, there exists in $G'_0$ a point $y$ such that $d_{G'_0}(y) = \delta(G'_0) \leq 3$. Let again $G''_0(4, q'') = G'_0(5, q') \setminus \{y\}$. One obtains then the following inequalities:

$$6 = q'' = q' - d_{G'_0}(y) \geq q' - 3 = (q - d_{G_0}(x)) - 3 \geq q - 4 - 3 \geq 13 - 7 = 6.$$

Thus $q'' = 6$, $d_{G''_0}(y) = \delta(G'_0) = 3$, $d_{G'_0}(x) = \delta(G_0) = 4$ and $q = 13$. Now since $d_{G_0}(y) = 3$ and $d_{G_0}(y) \geq 4$, it follows that $y$ is joined by an arc to (or from) $x$ in $G_0$, and $d_{G_0}(y) = d_{G_0}(x) = 4$. We shall prove that no graph $G(6, 13)$ obtained from $G''(4, 6)$ by adding two points $x$ and $y$ joined by an arc such that $d_{G}(x) = d_{G}(y) = 4$ can belong to $\mathcal{G}_3$. There are three graphs $G''(4, 6)$ in $\mathcal{G}_3$, $i = 1, 2, 3$ (see Fig. 8).

![Fig. 8.](image)

Each contains a point $d_i$ from which there are three distinct paths to another point $a_i$, $i = 1, 2, 3$. Since $\delta(G) = 4$, this implies that $d_i$ is adjacent to (or from) $x$ or $y$ in $G(6, 13)$. It is, however, easy to see that the graph $G(6, 13)$ cannot belong to $\mathcal{G}_3$, a contradiction. Thus $G(6, q) \in \mathcal{G}_3$ always satisfies the inequality $q \leq 12$, i.e. the relation (R).
Now we are able to finish the proof of theorem 1.1. We have shown that all \( G(p, q) \) from \( \mathcal{G}_3 \) satisfy (R) for \( p = 6 \). Let \( G(n, q) \in \mathcal{G}_3 \) for \( n > 6 \) and assume this assertion is true for all graphs \( G(p, q) \in \mathcal{G}_3 \) for which \( p \leq n - 1 \). By lemma 1, there is a point \( x \) in \( G \) such that \( d_G(x) \leq \left\lfloor \frac{1}{4}(p + 1) \right\rfloor \).

Since \( G' = G \setminus \{x\} \) is in \( \mathcal{G}_3 \), it satisfies (R) by the induction hypothesis. Therefore, \( G \) satisfies (R) by lemma 2. Hence

\[
f_3(p) \leq 2(p - 2) + \left\lfloor \frac{1}{4}(p - 2)^2 \right\rfloor.
\]

However, the complete tripartite graph \( (A, B, C, U) \) with orientation from \( A \) to \( B \), from \( A \) to \( C \) and from \( C \) to \( B \) where \( |A| = \left\lfloor \frac{1}{4}(p - 2) \right\rfloor \), \( |B| = \left\lfloor \frac{1}{4}(p - 2) \right\rfloor \) and \( |C| = 2 \) clearly belongs to \( \mathcal{G}_3 \) and

\[
|U| = q = 2(p - 2) + \left\lfloor \frac{1}{4}(p - 2)^2 \right\rfloor.
\]

Therefore,

\[
f_3(p) = 2(p - 2) + \left\lfloor \frac{1}{4}(p - 2)^2 \right\rfloor.
\]

2. CHARACTERIZATIONS OF EXTREMAL GRAPHS

In [1], we have found that for \( p \geq 4 \),

\[
f_1(p) = \left\lfloor \frac{1}{4}p^2 \right\rfloor,
\]

\[
f_2(p) = \left\lfloor \frac{1}{4}(p - 1) \right\rfloor + \left\lfloor \frac{1}{4}p^2 \right\rfloor;
\]

the result of the previous section was that

\[
f_3(p) = 2(p - 2) + \left\lfloor \frac{1}{4}(p - 2)^2 \right\rfloor \quad \text{for} \quad p \geq 6
\]

where

\[
f_k(p) = \max \{ q \mid G(p, q) \in \mathcal{G}_k \}, \quad k = 1, 2, 3.
\]

In this section, we shall give characterizations of all graphs \( G(p, f_k(p)) \) in \( \mathcal{G}_k \), \( k = 1, 2, 3 \).

**Theorem 2.1.** Every graph \( G(p, f_1(p)) \) in \( \mathcal{G}_1 \), \( p \geq 5 \), is a complete bipartite graph \( (A, B, U) \) with arcs oriented from \( A \) to \( B \) where either \( |A| = \left\lfloor \frac{1}{4}p \right\rfloor \) and \( |B| = \left\lfloor \frac{1}{4}p \right\rfloor \), or \( |A| = \left\lfloor \frac{1}{4}p \right\rfloor \) and \( |B| = \left\lfloor \frac{1}{4}p \right\rfloor + 1 \). These two cases are distinct iff \( p \) is odd.

**Proof.** We shall first establish three lemmas. For brevity, we call \( \mathcal{G}_1^m \) the set of all graphs \( G(p, f_1(p)) \), \( p \geq 5 \).

**Lemma 3.** No graph \( G \in \mathcal{G}_1^m \) with \( p \geq 5 \) points contains any cycle.

**Proof.** Let \( G_0(p, f_1(p)) \in \mathcal{G}_1 \) contain a cycle \( C_n \) of length \( n \geq 3 \). Contracting \( C_n \) to a single point, we obtain a graph \( G(p - n + 1, f_1(p) - n) \) which again belongs to \( \mathcal{G}_1 \). Hence

\[
\left\lfloor \frac{1}{4}p^2 \right\rfloor - n \leq \left\lfloor \frac{1}{4}(p - n + 1)^2 \right\rfloor,
\]

a contradiction since (1) is not true for \( p \geq 5 \) and \( n \geq 3 \).
Remark. It is easy to see that for $p = 4$, the only graphs in $\mathcal{G}_1^m$ which contain cycles are those in Fig. 9.

**Lemma 4.** If $G(p, f_1(p)) \in \mathcal{G}_1$, $p \geq 5$ then $\delta(G) \geq \lceil \frac{1}{4}p \rceil$.

**Proof.** Since for any point $x$ of $G$ the graph $G' = G \setminus \{x\}$ belongs to $\mathcal{G}_1$, we have
\[ \lceil \frac{1}{4}p^2 \rceil - d_o(x) \leq \lceil \frac{1}{4}(p - 1)^2 \rceil. \]
Hence
\[ d_o(x) \geq \lceil \frac{1}{4}p^2 \rceil - \lceil \frac{1}{4}(p - 1)^2 \rceil = \frac{1}{4}p \]
so that
\[ \delta(G) \geq \frac{1}{4}p. \]

**Lemma 5.** No graph in $\mathcal{G}_1^m$ with $p \geq 5$ points contains a directed path of length greater than one.

**Proof.** Let $G_0(p, f_1(p))$ belonging to $\mathcal{G}_1^m$ contain a directed path $L_m$ of length $m \geq 2$. Any point in $G_0$ which does not belong to $L_m$ is joined by an arc with at most one point of $L_m$ by ($P_1$) and lemma 3.

From this and (2), we obtain the double inequality
\[ (m + 1) \lceil \frac{1}{4}p \rceil \leq \sum_{i=1}^{m+1} d(x_i) \leq p - m - 1 + 2(m - 1) + 2. \]
Hence
\[ (m + 1) \lceil \frac{1}{4}p \rceil \leq p + m - 1. \]
A simple calculation shows that (4) is not true for \( p > 5 \) and \( m \leq 2 \). For \( p = 5 \), (4) implies \( m \leq 2 \). Since \( m \leq 2 \), we have to consider only the case \( p = 5 \) and \( m = 2 \). The inequality (3) yields then the equality

\[
3\left[\frac{3}{2}\right] = 6 = \sum_{i=1}^{3} d(x_i) = 6
\]

which implies \( d(x_i) = 2 \), \( i = 1, 2, 3 \). The corresponding graph is that in Fig. 10 which is a contradiction since \( q = 5 < f_1(5) = 6 \).

Let us finish now the proof of theorem 2.1. Let \( G \in \mathcal{G}_1^m \); By lemma 5, \( G \) contains no directed path of length greater than one. It follows easily that \( G \) is a bipartite graph \((A, B, U)\) with arcs oriented, say, from \( A \) to \( B \). If \( |A| = t \) then \( |B| = p - t \) so that

\[
q = \left\lfloor \frac{1}{2} p^2 \right\rfloor \leq t(p - t).
\]

This implies easily that \( t \geq \lceil \frac{1}{2} p \rceil \) or \( t = \lfloor \frac{1}{2} p \rfloor \). The proof is complete.

**Theorem 2.2.** Every graph \( G(p, f_2(p)) \) in \( \mathcal{G}_2 \) with \( p \geq 6 \) is a complete tripartite graph \((A, B, C, U)\) with arcs oriented from \( A \) to \( B \), from \( A \) to \( C \) and from \( C \) to \( B \) where either \( |A| = \lceil \frac{1}{2}(p - 1) \rceil, |C| = 1 \) and \( |B| = \lfloor \frac{1}{2}(p - 1) \rfloor \), or \( |A| = \lfloor \frac{1}{2}(p - 1) \rfloor, |C| = 1 \) and \( |B| = \lceil \frac{1}{2}(p - 1) \rceil \). These two cases are distinct iff \( p \) is even.

**Proof.** We shall prove first two lemmas.

**Lemma 6.** For any graph \( G(p, f_2(p)) \) in \( \mathcal{G}_2 \), \( p \geq 6 \), we have \( \delta(G) = \lfloor \frac{1}{2} p \rfloor \) and if \( y \) is a point in \( G \) for which \( f_2(y) = \lfloor \frac{1}{2} p \rfloor \) then the graph \( G' = G \setminus \{y\} \) is in \( \mathcal{G}_2 \) as well.

**Proof.** If \( x \) is any point of a graph \( G \in \mathcal{G}_2 \), \( G' = G \setminus \{x\} \) belongs to \( \mathcal{G}_2 \) so that

\[
f_2(p) - d_G(x) \leq f_2(p - 1).
\]

This implies

\[
d_G(x) \geq p - 1 + \left[\frac{1}{2}(p - 1)^2\right] - (p - 2) - \left[\frac{1}{2}(p - 1)^2\right] = \lfloor \frac{1}{2} p \rfloor.
\]

Hence

\[
\delta(G) \geq \lfloor \frac{1}{2} p \rfloor.
\]

However, in [1] we have proved that

\[
\delta(G) \leq \lfloor \frac{1}{2} p \rfloor.
\]

Thus

\[
\delta(G) = \lfloor \frac{1}{2} p \rfloor.
\]

On the other hand, let a point \( y \) of \( G(p, f_2(p)) \in \mathcal{G}_2 \) satisfy \( d_G(y) = \lfloor \frac{1}{2} p \rfloor \); the above calculation shows that

\[
f_2(p) - d_G(y) = f_2(p - 1),
\]

i.e. that \( G' = G \setminus \{y\} \in \mathcal{G}_2 \).
Lemma 7. $\mathcal{G}_2^p$ contains only two graphs of order $p = 6$. Both are complete tripartite graphs $(A, B, C, U)$ with arcs oriented from $A$ to $B$, from $A$ to $C$ and from $C$ to $B$. For one of them, $|A| = 3$, $|B| = 2$, $|C| = 1$, for the other $|A| = 2$, $|B| = 3$, $|C| = 1$.

Proof. We have $f_2(6) = 11$. Let thus $G(6, 11) \in \mathcal{G}_2^p$. By lemma 6, $\delta(G) = 3$ and if a point $x$ of $G$ satisfies $d_G(x) = 3$ then the graph $G'(5, 8) = G(6, 11) \setminus \{x\}$ is in $\mathcal{G}_2^p$ and $\delta(G') = 3$. The only possible distribution of degrees of vertices in $G'(5, 8)$ is $(4, 3, 3, 3, 3)$ and that of $G(6, 11)$ is $(4, 4, 4, 3, 3, 3)$. It follows easily that then $G$ with deleted orientation is the graph in Fig. 11. Denote by $G_0(6, 11)$ and $G_0'(5, 8) = G_0(5, 8) \setminus \{x\}$ these non-oriented graphs. We shall investigate the possible orientations of the arcs of $G_0$ and deduce then those of $G_0$. Without difficulty one finds that the only possible orientations of $G_0'$ are those two in Fig. 12 (any other orientation gives more than two directed paths from one point to another). While the graph (a) does not create any possible graph $G_0$, the graph (b) leads to the two described in the lemma.

To complete the proof of theorem 2.2, we shall use induction with respect to $p$. For $p = 6$, the theorem is true by lemma 7. Suppose that $p \geq 6$ and that the theorem is true for all graphs in $\mathcal{G}_2^p$ of order $p$. Let $G(p + 1, f_2(p + 1)) \in \mathcal{G}_2^p$. By lemma 6, there is a point $x$ of $G$ with the minimum degree $\lceil \frac{p}{2} \rceil$. The graph $G' = G \setminus \{x\}$
has order $p$ and since it belongs to $\mathcal{G}_2^p$ by lemma 6, it is a complete bipartite graph $(A_1, B_1, C_1, U_1)$ oriented from $A_1$ to $B_1$, from $A_1$ to $C_1$ and from $C_1$ to $B_1$, with either

$$|A_1| = \left\lfloor \frac{1}{2} (p - 1) \right\rfloor, \quad |B_1| = \left\lfloor \frac{1}{2} (p - 1) \right\rfloor, \quad |C_1| = 1 \quad (\text{case 1})$$

or

$$|A_1| = \left\lfloor \frac{1}{2} (p - 1) \right\rfloor, \quad |B_1| = \left\lceil \frac{1}{2} (p - 1) \right\rceil \quad \text{and} \quad |C_1| = 1 \quad (\text{case 2}).$$

Let us notice first that there cannot be any arc to $x$ from a point of $A_1$; since there must be at least one more arc to or from $x$ from or to $C_1$ or $B_1$, one gets four cases

Let us notice first that there cannot be any arc to $x$ from a point of $A_1$; since there must be at least one more arc to or from $x$ from or to $C_1$ or $B_1$, one gets four cases.
in Fig. 13-a. Each of them leads to a contradiction. Similarly, there cannot be any arc to \( x \) from a point in \( B_1 \) by Fig. 13-b and 13-a. There is also no pair of arcs from \( A_1 \) to \( x \) and from \( x \) to \( B_1 \) (Fig. 14), no pair of arcs from \( A_1 \) to \( x \) and from \( x \) to \( C_1 \) (Fig. 15(a)) as well as no pair of arcs from \( C_1 \) to \( x \) and from \( x \) to \( B_1 \) (Fig. 15(b)).

![Fig. 14.](image)

![Fig. 15.](image)

It follows that either there are arcs to \( x \) from all the points in \( A_1 \cup C_1 \) in case 2, or there are arcs from \( x \) to all points in \( C_1 \cup B_1 \) in case 1. In each case, one obtains that \( G \) is a tripartite graph of order \( p + 1 \) which satisfies the conditions in the theorem. The rest is obvious.

**Theorem 2.3.** Every graph \( G(p,f_3(p)) \) in \( \mathcal{G}_3^m \) with \( p \geq 7 \) is a complete tripartite graph \((A, B, C, U)\) with arcs oriented from \( A \) to \( B \), from \( A \) to \( C \) and from \( C \) to \( B \), where either \( |A| = \lfloor \frac{1}{3}(p - 2) \rfloor, |B| = \lceil \frac{2}{3}(p - 2) \rceil, |C| = 2 \), or \( |A| = \lceil \frac{2}{3}(p - 2) \rceil, |B| = \lfloor \frac{1}{3}(p - 2) \rfloor, |C| = 2 \). Both cases coincide iff \( p \) is even.

**Proof.** We shall first prove a lemma.

**Lemma 8.** For any graph \( G(p,f_3(p)) \) in \( \mathcal{G}_3^m \), \( p \geq 7 \), we have \( \delta(G) = \lceil \frac{1}{3}(p + 1) \rceil \). If \( y \) is a point of \( G \) such that \( d_G(y) = \lfloor \frac{1}{3}(p + 1) \rfloor \) then the graph \( G' = G \setminus \{y\} \) belongs again to \( \mathcal{G}_3^m \).

**Proof.** If \( x \) is any point of a graph \( G \in \mathcal{G}_3^m \) with \( p \geq 3 \) points, then \( G' = G \setminus \{x\} \) is in \( \mathcal{G}_3 \). Hence

\[
f_3(p) - d_G(x) \leq f_3(p - 1)
\]
which implies
\[ d_\alpha(x) \geq f_3(p) - f_3(p - 1) = \lceil \frac{p}{4} \rceil. \]

Thus
\[ \delta(G) \geq \lceil \frac{p}{4} \rceil. \]

By lemma 1,
\[ \delta(G) \leq \lceil \frac{p}{4} \rceil. \]

so that
\[ \delta(G) = \lceil \frac{p}{4} \rceil. \]

If a point \( y \) in \( G \) satisfies \( d_\alpha(y) = \lceil \frac{p}{4} \rceil \), the above calculation shows that
\[ f_3(p) - d_\alpha(y) = f_3(p - 1), \]

i.e. that \( G' = G \setminus \{y\} \) belongs to \( \mathcal{G}_3^m \).

Returning to the proof of theorem 2.3, we shall investigate graphs in \( \mathcal{G}_3^m \) with six points. Let \( G(6, 12) \) be such a graph. By lemma 1, \( \delta(G) \leq 4 \). Let \( x \) be a point from \( G(6, 12) \) for which \( d_\alpha(x) = \delta(G) \) and let \( G'(5, q') = G(6, 12) \setminus \{x\} \). Since \( G'(5, q') \in \mathcal{G}_3 \), \( \delta(G') \leq 3 \) by lemma 1. Let \( y \) be a point from \( G'(5, q') \) of the smallest degree, i.e. \( d_\alpha(y) = \delta(G') \). If \( G''(4, q'') = G' \setminus \{y\} \) then
\[ 6 \geq q'' = q' - \delta(G') = (q - \delta(G)) - \delta(G') = 12 - \delta(G) - \delta(G') \]

so that
\[ \delta(G) + \delta(G') \geq 6. \]

But
\[ \delta(G) + \delta(G') \leq 4 + 3 = 7. \]

We have thus two cases:

Case A. \( \delta(G) = 4 \) and \( \delta(G') = 3 \).

Then the only point \( y \) in \( G' \) which is not joined with \( x \) by an arc (of any orientation) has degree four. The distribution of the degrees in \( G' \) is thus \( (4, 3, 3, 3, 3) \), that in \( G \) is \( (4, 4, 4, 4, 4, 4) \). The graph \( G \) with omitted orientation is given in Fig. 16. It is not difficult to see that the subgraph \( G' \) of \( G \) cannot contain three directed paths from
a point of degree three to another point of degree three. Therefore, the only graphs which can serve as $G'$ are those in Fig. 17. Here, the graph in Fig. 17(c) cannot be completed into $G$ in $\mathcal{G}_3^n$ while the remaining two can be completed in a single way, both resulting in the graph $G_1(6, 12)$, the complete tripartite graph $(A, B, C, U)$ with arcs oriented from $A$ to $B$, from $A$ to $C$ and from $C$ to $B$, $|A| = |B| = |C| = 2$.

![Fig. 17.](image)

Case B. $\delta(G) = \delta(G') = 3$.

The distribution of the degrees in $G'(5,9)$ will then be $(4, 4, 4, 3, 3)$ and $G'$ has as its subgraphs one of the graphs in Fig. 8. Let us denote in each of them (Fig. 18) one or more points as $d_i$ (departure) and one or more points as $a_j$ (arrival) in such a way that there are exactly three directed paths from each $d_i$ to each $a_j$. Thus, for $i = 1, 2, 3$, $G_i'' = G'(5,9) \setminus \{y\}$ where $y$ is a point joined with three points of $G_i'$. Observe that if $y$ is adjacent to or from a point $d_k$ then the only possibility is that $y$ is adjacent from two points different from the $a_j$'s. Similarly, if $y$ is adjacent to or from a point $a_k$ then the only possibility is that $y$ is adjacent to two points different from the $d_i$'s. It is easily seen that no graph $G'(5,9)$ contains the graph $G_1''$ as subgraph and that only the graphs $G_2'', G_3''$ in Fig. 19 contain $G_2'', G_3''$, respectively. (Observe that $G_2'', G_3''$ as well as $G_2', G_3'$ arise from each other by change of orientation of arcs.) An analogue reasoning allows to construct the graphs $G(6,12)$ using the graphs

![Fig. 18.](image)
Let us turn now to graphs in \( \mathcal{F}_3 \) with seven points and prove the theorem holds in this case. Let \( G_0(7, 16) \) be a graph in \( \mathcal{F}_3 \). By lemma 8, \( \delta(G_0) = 4 \) and if for a point \( x_0 \), \( d_{G_0}(x_0) = 4 \) then \( G(6, 12) = G_0(7, 16) \setminus \{x_0\} \) is in \( \mathcal{F}_3 \). One can thus use the described technique again. We obtain the graphs in Fig. 21 from the graph \( G_1(6, 12) \). These are both complete tripartite as asserted. (The graphs \( G_2(6, 12) \) and \( G_3(6, 12) \) do not yield any graph \( G_0(7, 16) \) since \( d_{G_0}(x_0) = 4 \) on one side and on the other side \( x_0 \) cannot be adjacent to or from both a point \( d_i \) and \( a_j \).)
We shall now complete the proof by induction with respect to $p$. We proved the theorem is true for $p = 7$. Thus assume a graph $G(p + 1, f_3(p + 1))$ belongs to $\mathcal{G}_3$ while the theorem is true for the graphs in $\mathcal{G}_3^{p}$ with $p \geq 7$ points. By lemma 8, there is a point $x$ in $G$ which has the minimum degree $\lceil \frac{p}{4} + 2 \rceil$. Moreover, the graph $G' = G \setminus \{x\}$ also belongs to $\mathcal{G}_3^{p}$ by the same lemma. By the induction hypothesis, $G'$ is a complete tripartite graph $(A_1, B_1, C_1, U_1)$ oriented from $A_1$ to $B_1$, from $A_1$ to $C_1$ and from $C_1$ to $B_1$ and such that $|A_1| = \lceil \frac{p}{4} - 2 \rceil$, $|B_1| = \lceil \frac{p}{4} - 2 \rceil$, $|C_1| = 2$ (case 1), or, if $p$ is odd, $|A_1| = \lceil \frac{3(p - 2)}{4} \rceil$, $|B_1| = \lceil \frac{3(p - 2)}{4} \rceil$, $|C_1| = 2$ (case 2).

Fig. 22.
Notice that there cannot be an arc from $x$ to a point in $A_t$ (Fig. 22) since there is always another arc incident with $x$ with the other end-point in $C_1$ or $B_1$. Similarly, there cannot be an arc from a point in $B_1$ to $x$ (Fig. 23 and Fig. 22d). There is also no pair of arcs from a point in $A_1 \cup C_1$ to $x$ and from $x$ to a point in $B_1$ (Fig. 24, 25) and similarly, no pair of arcs from $A_1$ to $x$ and from $x$ to $B_1 \cup C_1$ (Fig. 24, 26).

Therefore, $x$ is either adjacent from all the points of $A_1 \cup C_1$ in case 1 if $p$ is even and in case 2, or $x$ is adjacent to all the points of $B_1 \cup C_1$ in case 1. In other cases, the degree of $x$ would not be minimal. The graph $G$ is then again a tripartite graph with the properties described in the theorem. The proof is complete.

References


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