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A SHARPENING OF DISCRETE ANALOGUES OF WIRTINGER’S INEQUALITY

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Discrete analogues of Wirtinger’s inequality have been already studied by various authors (see e.g., [1], [2], [3], [5], [6], [7], [8]). Z. Nádeník in [4] proved the following sharpening of Wirtinger’s inequality:

**Theorem 1.** Let \( f(\varphi) \) denote a continuous function with the period \( 2\pi \), which has the symmetrical derivative \( f^*(\varphi) = \lim_{\varepsilon \to 0} \frac{f(\varphi + \varepsilon) - f(\varphi - \varepsilon)}{2\varepsilon} \): (2\varepsilon). Let \( f^*(\varphi) \) be of bounded variation in \((0, 2\pi)\). If

\[
\int_0^{2\pi} f(\varphi) \, d\varphi = 0 ,
\]

then

\[
\int_0^{2\pi} f^{*2}(\varphi) \, d\varphi - \int_0^{2\pi} f^2(\varphi) \, d\varphi - \frac{\pi}{2} [f(0) + f(\pi)]^2 \geq 0
\]

with the equality holding only for

\[
f(\varphi) = a \cos \varphi + b \sin \varphi + c(2 - \pi |\sin \varphi|) , \quad a, b, c = \text{const.}
\]

In this paper we prove sharpenings of two discrete analogues of Wirtinger’s inequality, which are analogous to Theorem 1 (Theorems 2, 3), using real trigonometric polynomials. Then we show a geometrical application — a sharpening of the isoperimetric inequality for some polygons (Theorem 4).

1. LIST OF THEOREMS

**Theorem 2.** Let \( n = 2m \), let \( x_1, \ldots, x_n \) be \( n \) real numbers such that

\[
\sum_{i=1}^{n} x_i = 0 .
\]
Let us define $x_{n+1} = x_1$. Then

$$
(1.2) \quad \sum_{i=1}^{n} (x_i - x_{i+1})^2 \geq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 + n \sin \frac{\pi}{n} \left( \sin \frac{2\pi}{n} - \sin \frac{\pi}{n} \right) (x_m + x_{2m})^2.
$$

The equality in (1.2) holds if and only if

$$
(1.3) \quad x_i = A \cos \frac{2\pi i}{n} + B \sin \frac{2\pi i}{n}, \quad i = 1, \ldots, n, \quad A, B = \text{const}.
$$

**Theorem 3.** Let $x_1, \ldots, x_n$ be $n$ real numbers satisfying (1.1), $n \geq 2$. Then

$$
(1.4) \quad \sum_{i=1}^{n} (x_i - x_{i+1})^2 \geq 4 \sin^2 \frac{\pi}{n} \sum_{i=1}^{n} x_i^2 +
$$

$$
+ 2n \sin \frac{\pi}{2n} \left( \sin \frac{\pi}{n} - \sin \frac{\pi}{2n} \right) (x_1 + x_n)^2.
$$

The equality in (1.4) holds if and only if

$$
(1.5) \quad x_i = A \cos \frac{(2i-1)\pi}{2n}, \quad i = 1, \ldots, n, \quad A = \text{const}.
$$

**Theorem 4.** Let $n = 2m$. Let $P = A_1 \ldots A_n$ denote an equilateral closed $n$-gon in $E_2$ of area $F$ and perimeter $L$. Let us denote by $d_i$ the distance of the center of $A_iA_{i+m}$ and the centroid of $P$, $d = \max \{d_1, \ldots, d_m\}$. Then

$$
(1.6) \quad L^2 \geq 4n \tan \frac{\pi}{n} F + 2n^2 \tan^2 \frac{\pi}{n} \left( 2 \cos \frac{\pi}{n} - 1 \right) d^2
$$

with the equality holding only for a regular $n$-gon.

2. **NOTATIONS AND AUXILIARY THEOREMS**

Let $n = 2m$. In [1] it is shown that there exist numbers $c_k, c_k^*, k = 0, \ldots, m$, $l = 1, \ldots, m - 1$, such that

$$
(2.1) \quad \sum_{i=1}^{n} x_i^2 = \frac{n}{2} \sum_{k=1}^{m-1} (c_k^2 + c_k^{*2}) + nc_m^2;
$$

$$
(2.2) \quad \sum_{i=1}^{n} (x_i - x_{i+1})^2 = 2n \sum_{k=1}^{m-1} (c_k^2 + c_k^{*2}) \sin^2 \frac{k\pi}{n} + 4nc_m^2,
$$
\[(x_m + x_{2m})^2 = 4 \left( \sum_{i=1}^{M} c_{2i} \right)^2, \text{ where } M = \lfloor n/4 \rfloor.\]

**Remark.** Recall that \([a] = a\) for \(a\) being an integer, \([a] = b\), where \(b\) is the biggest integer smaller than \(a\), otherwise.

Let us denote
\[(2.4)\quad A(n) = n \sin \frac{\pi}{n} \left( \sin \frac{2\pi}{n} - \sin \frac{\pi}{n} \right).\]

By virtue of (2.1)–(2.4) the inequality (1.2) can be written as
\[
\sum_{k=1}^{m-1} \left( c_k^2 + c_k^{'2} \right) \left( \sin^2 \frac{k\pi}{n} - \sin^2 \frac{\pi}{n} \right) + 2c_m^2 \left( 1 - \sin^2 \frac{\pi}{n} \right) - \frac{2}{n} A(n) \left( \sum_{i=1}^{M} c_{2i} \right)^2 \geq 0.
\]

The following inequalities hold:
\[(2.5)\quad \sin^2 \frac{k\pi}{n} - \sin^2 \frac{\pi}{n} < 0, \quad k = 1,$$
\[
\sin^2 \frac{\pi}{n} > 0, \quad k = 2, \ldots, m - 1,$$
\[(2.6)\quad \sin^2 \frac{\pi}{n} < 1.
\]

Using (2.5), (2.6) we conclude that it is sufficient to prove that
\[(2.7)\quad L = \sum_{k=1}^{M} c_{2k}^2 \left( \sin^2 \frac{2k\pi}{n} - \sin^2 \frac{\pi}{n} \right) - \frac{2}{n} A(n) \left( \sum_{k=1}^{M} c_{2k} \right)^2 \geq 0.
\]

**Lemma 1.** Let us denote
\[C_1(n) = 2 \sin \frac{\pi}{n} \left( \sin \frac{2\pi}{n} + \sin \frac{\pi}{n} \right),
\]
\[K_i(n) = \sin^2 \frac{2(i + 1)\pi}{n} - \sin^2 \frac{\pi}{n}, \quad i = 1, \ldots, M - 1.
\]

Then
\[(2.8)\quad C_1(n) \sum_{i=1}^{r} \frac{1}{K_i(n)} < 1, \quad r = 1, \ldots, M - 1.
\]

**Proof.** Clearly it is sufficient to prove (2.8) for \(r = M - 1\). Denote
\[f(n) = C_1(n) \sum_{i=1}^{M-1} \frac{1}{K_i(n)}.
\]

We have to show that \(f(n) < 1\).

We can suppose \(n \geq 8\). Then
\[K_i(n) = \sin \frac{2i + 3}{n} \pi \sin \frac{2i + 1}{n} \pi.
\]
We have
\[ \cotg \alpha - \cotg (\alpha + \beta) = \frac{\sin \beta}{\sin \alpha \sin (\alpha + \beta)} \]
and therefore
\[ \left( \text{for } \alpha = \frac{2i + 1}{n} \pi, \beta = \frac{2}{n} \pi, \text{ i.e. } \alpha + \beta = \frac{2i + 3}{n} \pi \right) \]
\[ \frac{1}{K_i(n)} = \frac{1}{\sin \frac{2\pi}{n}} \left( \cotg \frac{2i + 1}{n} \pi - \cotg \frac{2i + 3}{n} \pi \right). \]

After adding these expressions for \( i = 1, \ldots, M - 1 \) we get
\[ \sum_{i=1}^{M-1} \frac{1}{K_i(n)} = \begin{cases} \frac{1}{\sin \frac{2\pi}{n}} \left( \cotg \frac{3\pi}{n} + \tan \frac{\pi}{n} \right) & \text{for } n = 4M, \\ \frac{1}{\sin \frac{2\pi}{n}} \cotg \frac{3\pi}{n} & \text{for } n = 4M + 2. \end{cases} \]

For \( n = 4M \) we conclude (by virtue of the identity \( \sin 2\alpha = 2 \sin \alpha \cos \alpha \))
\[ f(n) = \frac{\cos \frac{\pi}{2n} \cos \frac{2\pi}{n}}{\cos^2 \frac{\pi}{2n} \cos \frac{3\pi}{2n}}. \]

Introducing the notation
\[ x = \frac{\pi}{n}, \quad y = \cos^2 \frac{x}{2}, \text{ i.e. } y = \cos^2 \frac{\pi}{2n}, \]
we obtain
\[ n \geq 8 \Rightarrow x \in (0, \pi/8), \quad y \in I = \left( \cos^2 \frac{\pi}{16}, 1 \right). \]

Substituting \( x = \pi/n \) in \( f(n) \) we get the function
\[ f_1(x) = \frac{\cos x}{2} \frac{\cos 2x}{\cos^2 x \cos \frac{3x}{2}}, \]
which is defined in $I$. Using the identities
\[
\cos x = 2y - 1, \\
\cos 2x = 2(2y - 1)^2 - 1, \\
\cos \frac{3x}{2} = \cos \frac{x}{2} (4y - 3),
\]
we get the function
\[
g(y) = \frac{8y^2 - 8y + 1}{(2y - 1)^2 (4y - 3)}.
\]
It is easy to show that $g(y) < 1$ in $I$ and therefore $f(n) < 1$ for $n = 4M, n \geq 8$.

Analogously, when considering $n = 4M + 2, n \geq 10$, we can show that
\[
f(n) = \frac{\cos \frac{\pi}{2n} \cos \frac{3\pi}{n}}{\cos \frac{\pi}{2n} \cos \frac{3\pi}{n}}
\]
and with $x = \pi/n, y = \cos^2 (x/2)$ we get the function
\[
h(y) = \frac{16y^2 - 16y + 1}{4y - 3},
\]
y $\in J = \left(\cos^2 \frac{\pi}{20}, 1\right)$.

Using the inequality $h(y) < 1, y \in J$, we conclude that $f(n) < 1$ for $n = 4M + 2, n \geq 10$.

So, Lemma 1 holds.

**Lemma 2.** Using the notation from Lemma 1, define
\[
C_{r+1}(n) = \frac{C_1(n)}{1 - C_1(n) \sum_{i=1}^{r} \frac{1}{K_i(n)}}, \quad r = 1, ..., M - 1.
\]

Then
\[
K_r(n) - C_r(n) > 0, \quad r = 1, ..., M - 1.
\]

**Proof.** It is easy to show that (2.9) holds for $r = 1$. Let $r \geq 2$. It can be shown that
\[
K_r(n) - C_r(n) = \frac{K_r(n) \left[ 1 - C_1(n) \sum_{i=1}^{r} \frac{1}{K_i(n)} \right]}{1 - C_1(n) \sum_{i=1}^{r-1} \frac{1}{K_i(n)}}.
\]
Now, (2.8) implies (2.9) for arbitrary $r = 1, ..., M - 1$. 
Lemma 3. For $r = 1, \ldots, M$, we have

\begin{equation}
L = \sum_{k=1}^{r} [c_{2k} D(k, n) - 2B(k, n) \sum_{i=k+1}^{M} c_{2i}]^2 + \left( \sum_{k=r+1}^{M} \left( \sin^2 \frac{2k\pi}{n} - \sin^2 \frac{\pi}{n} \right) c_{2k}^2 - C(r, n) \left( \sum_{k=r+1}^{M} c_{2k} \right)^2 \right)
\end{equation}

where

\begin{equation}
C(1, n) = 2 \sin \frac{\pi}{n} \left( \sin \frac{2\pi}{n} + \sin \frac{\pi}{n} \right) \left[ = C_1(n) \right],
\end{equation}

\begin{equation}
B(1, n) = \sin \frac{\pi}{n},
\end{equation}

\begin{equation}
D(1, n) = \sin \frac{2\pi}{n} - \sin \frac{\pi}{n},
\end{equation}

\begin{equation}
D^2(k + 1, n) = \sin^2 \frac{2(k + 1)\pi}{n} - \sin^2 \frac{\pi}{n} - C(k, n),
\end{equation}

\begin{equation}
B(k + 1, n) = \frac{C(k, n)}{2D(k + 1, n)},
\end{equation}

\begin{equation}
C(k + 1, n) = C(k, n) + 4B^2(k + 1, n),
\end{equation}

$k = 1, \ldots, M - 1$.

Remark. We have to verify that the definition of the numbers $D(k, n), B(k, n), C(k, n)$ in (2.11) is correct, i.e. that

\[ K_k(n) - C(k, n) > 0, \quad k = 1, \ldots, M - 1. \]

We shall show that $C(k, n) = C_k(n)$, $C_k(n)$ being the numbers defined in Lemma 2. It is true for $r = 1$ [see (2.11)]. Let $C(k, n) = C_k(n)$ for an integer $k$, $1 \leq k < M - 1$. Then $D^2(k + 1, n) = K_k(n) - C_k(n) > 0$ and therefore $C(k + 1, n)$ is defined in (2.11). It is easy to show that

\[ C(k + 1, n) = \frac{C_1(n)}{1 - C_1(n) \sum_{i=1}^{k} \frac{1}{K_i(n)}} = C_{k+1}(n). \]

The inequality $K_k(n) - C(k, n) > 0$ is now a consequence of Lemma 2.

Proof. Let us denote by $L_i$ the representation of $L$ in (2.10) for $r = i$. We have to show

\[ L_r = L_i, \quad r = 1, \ldots, M. \]
We use the induction over \( r \). Let \( r = 1 \). To prove that \( P = L_1 - L = 0 \) we write [see (2.11)]

\[
P = \left[ c_2 \left( \sin \frac{2\pi}{n} - \sin \frac{\pi}{n} \right) - 2 \sin \frac{\pi}{n} \sum_{i=2}^{M} c_{2i} \right]^2 - C_1(n) \left( \sum_{k=1}^{M} c_{2k} \right)^2 -
\]

\[
- c_2^2 \left( \sin^2 \frac{2\pi}{n} - \sin^2 \frac{\pi}{n} \right) + \frac{2}{n} A(n) \left( \sum_{k=1}^{M} c_{2k} \right)^2.
\]

Adding and subtracting \( (2/n) A(n) \left( \sum_{k=2}^{M} c_{2k} \right)^2 \) we get

\[
P = 2c_2^2 \left( \frac{A(n)}{n} + 2 \sin^2 \frac{\pi}{n} - \sin \frac{2\pi}{n} \sin \frac{\pi}{n} \right) +
\]

\[
+ 4c_2 \left( \sum_{k=2}^{M} c_{2k} \right) \left[ \frac{A(n)}{n} - \sin \frac{\pi}{n} \left( \sin \frac{2\pi}{n} - \sin \frac{\pi}{n} \right) \right] +
\]

\[
+ \left( \sum_{k=2}^{M} c_{2k} \right)^2 \left[ 4 \sin^2 \frac{\pi}{n} - C_1(n) + \frac{2}{n} A(n) \right].
\]

From (2.11) it follows that \( P > 0 \).

Let \( L_r = L \) for an integer \( r \), \( 1 = r < M \). We show that \( L_{r+1} = L \), i.e. \( R = L_{r+1} - L_r = 0 \). Analogously to the case \( r = 1 \) we get

\[
R = c_2^2 \left( r + 1, n \right) \left[ D^2(r + 1, n) - \sin^2 \frac{2(r + 1)\pi}{n} + \sin^2 \frac{\pi}{n} + C(r, n) \right] +
\]

\[
+ \left( \sum_{k=r+2}^{M} c_{2k} \right)^2 \left[ C(r, n) - C(r + 1, n) + 4B^2(r + 1, n) \right] +
\]

\[
+ c_2(r+1) \left( \sum_{k=r+2}^{M} c_{2k} \right) \left[ -4B(r + 1, n) D(r + 1, n) + 2C(r, n) \right].
\]

Using (2.11) we conclude \( R = 0 \).

So, (2.10) holds.

3. PROOFS OF THEOREMS

Theorem 2. The inequality (2.7) [and so (1.2) as well] is a consequence of Lemma 3. Choose \( r = M \). Then

\[
(3.1) \quad L = \sum_{k=1}^{M-1} \left[ c_{2k} D(k, n) - 2B(k, n) \sum_{i=k+1}^{M} c_{2i} \right]^2 + D^2(M, n) c_{2M}^2.
\]

According to (2.11) and (2.9) we conclude that

\[
(3.2) \quad L \geq 0.
\]
Conditions for the equality follow from (2.5), (2.6) and (3.1).

Theorem 3. If we use Theorem 2 for real numbers $y_1, \ldots, y_{2n}$ defined as follows

$$y_k = \begin{cases} x_k, & k = 1, \ldots, n, \\ x_{2n-k+1}, & k = n + 1, \ldots, 2n, \end{cases}$$

we get the required inequality. (See the proof of Theorem 4 in [6].)

Theorem 4. In [6], Section 4, the following is proved:

$$8 \tan \frac{\pi}{n} F = \sum_{i=1}^{n} \left(1 - \tan^{2} \frac{\pi}{n}\right) \left[(x_i - x_{i+1})^2 + (y_i - y_{i+1})^2\right] +$$

$$+ 4 \tan^{2} \frac{\pi}{n} \sum_{i=1}^{n} (x_i^2 + y_i^2),$$

where $A_i = [x_i, y_i], \ i = 1, \ldots, n$, in the coordinate system $S = \{O, x, y\}$ in $E_2$ with $O$ being the centroid of $P$. In the system $S$ the assumptions of Theorem 2 for \{x_i\}, \{y_i\} are satisfied. Hence Theorem 4 follows from Theorem 2.

References


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